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Extreme Quantum Advantage when Simulating Strongly Coupled Classical Systems

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Classical stochastic processes can be generated by quantum simulators instead of the more standard classical ones, such as hidden Markov models. One reason for using quantum simulators is that they generally require less memory than their classical counterparts. Here, we examine this quantum advantage for strongly coupled spin systems—the Dyson-like one-dimensional Ising spin chain with variable interaction length. We find that the advantage scales with both interaction range and temperature, growing without bound as interaction increases. Thus, quantum systems can very efficiently simulate strongly coupled classical systems.

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I. INTRODUCTION

We illustrate, what seems to be, an emerging principle relating the classical and quantum worlds: strongly correlated classical systems can be efficiently simulated by quantum systems. Moreover, we show that this quantum advantage is substantial and increases with the classical system’s degree of interaction.

Our results suggest that this principle may well lead to broad consequences. The world is nothing, if not its constituent interactions. Some go so far as to argue that *interaction*, not object, is the basic unit of reality [1, 2]. At a minimum, though, interactions make our world interesting and structured. They also complicate it, and rapidly as the web of interaction widens. The resulting complicatedness often confounds theory and drives us to resort to simulation.

Statistical mechanical spin systems are a prime example: A broadly used class of models that can be arbitrarily hard to analyze and, somewhat soberingly, even hard to simulate [3, 4]. While there is a plurality of complicated spin systems, we choose one that is sufficiently interesting, but also very familiar territory: the one-dimensional Ising spin chain with variable interaction range. The one-dimensionality provides tractability, while the long-range coupling yields the desired complicated configuration structure.

Selecting a system is only the start, of course. There are many different simulation methods for any given class. The differences range from the purely algorithmic to the physical substrate employed. For instance, a computer can use machine code or high-level programming lan-

guages to yield the desired result. Additionally, computers come in different varieties, in particular the substrate may be classical or quantum [5]. Just as the choice of code influences the computational resources required, so does the substrate.

Here, we report on the memory resources required by classical and quantum simulators. It is known that a quantum simulator typically requires less memory than its classical counterpart.¹ We refer to this as the *quantum advantage*. Despite exploring several particular cases, very little is known about how this quantum advantage scales. Addressing this, the following shows that not only is the advantage substantial, it also increases systematically and without bound.

To establish the scalings we compare classical and quantum simulators that generate spin configurations of the one-dimensional Ising model with N -nearest neighbor interactions. To compute the quantum advantage, we adapt the transfer-matrix formalism to construct the two simulators. This technique allows us to numerically (but accurately) determine the scaling behavior. We find that not only is the quantum advantage generic, its growth scales with temperature and interaction length.

II. DYSON-ISING SPIN CHAIN

We begin with a general one-dimensional ferromagnetic Ising spin chain [7, 8] with Hamiltonian:

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J(i,j) s_i s_j ,$$

¹ The memory resources required are equal if and only if the classical stochastic process is that with no crypticity [6].

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in contact with thermal bath at temperature T ,² where spin s_i at site i takes on values $\{+1, -1\}$ and $J(i, j) \geq 0$ is the spin coupling constant between sites i and j . Assuming translational symmetry, $J(i, j) \rightarrow J(k)$, $k \equiv |i - j|$. Commonly, $J(k)$ is a positive and monotone-decreasing function. An interaction is said to be *long-range* if $J(k)$ decays more slowly than exponential. In our studies, we consider couplings that decay by a power law:

$$J(k) = \frac{J_0}{k^\delta},$$

where $\delta > 0$. The spin chain resulting from these assumptions is called the *Dyson* model [9].

To approximate such a long-range system one can consider a similar system with finite-range interaction. For every interaction range N , we define the approximate Hamiltonian:

$$\mathcal{H}_N = - \sum_i \sum_{k=1}^N \frac{J_0}{k^\delta} s_i s_{i+k}.$$

This class of Hamiltonians can certainly be studied in its own right, not simply as an approximation. Let's explore its set of equilibrium configurations as a stochastic process.

III. PROCESSES

The concept of a stochastic process is very general. Any physical system that exhibits stochastic dynamics in time may be thought of as *generating* a stochastic process. We focus on *discrete-time, discrete-valued stationary stochastic processes*. Such a process, denoted $P = \{\mathcal{A}^\infty, \Sigma, \mathbb{P}(\cdot)\}$, is a probability space [10, 11]. Here, the observed symbols come from an alphabet $\mathcal{A} = \{\downarrow, \uparrow\}$ of local spin states; though our results easily extend to any finite alphabet. Each random spin variable X_i , $i \in \mathbb{Z}$, takes values in \mathcal{A} . $\mathbb{P}(\cdot)$ is the probability measure over the bi-infinite chain of random variables $X_{-\infty:\infty} = \dots X_{-2} X_{-1} X_0 X_1 X_2 \dots$. Σ is the σ -algebra generated by the cylinder sets in \mathcal{A}^∞ . *Stationarity* means that $\mathbb{P}(\cdot)$ is invariant under time translation. That is, $\mathbb{P}(X_{i_1} X_{i_2} \dots X_{i_m}) = \mathbb{P}(X_{i_1+n} X_{i_2+n} \dots X_{i_m+n})$, for all $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$.

To interpret our Ising system as a stochastic process, we consider not its time evolution, but rather the spatial “dynamic”. A spin configuration at one instant of

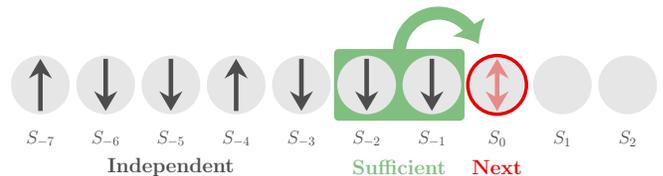


FIG. 1. A Markov order- N process generates a spin configuration from left-to-right. Markov order $N = 2$ shown. The values of an isolated spin S_0 , say, is undetermined. To make this (stochastic) choice consistent with the overall process and the particular instantiation on the left, it is sufficient to consider only the previous N (2) spins (highlighted in green).

time may be thought of as having been generated left-to-right (or equivalently right-to-left). The probability distribution over these configurations defines a stochastic process $P(N, T)$ that inherits its stationarity from spin-configuration spatial translation invariance. In this way, we build on earlier work that used computational mechanics to analyze statistical structure in spatial configurations generated by spin systems [12, 13].

Now that we have defined the process of interest, let us introduce two of its simulators.

IV. CLASSICAL AND QUANTUM SIMULATORS

What is a simulator for a stochastic process? Often, “simulation” refers to an approximation. In contrast, we require our simulators to be perfect, to generate P ’s configurations and their probabilities exactly. Our simulator, though, does more than correctly reproduce a probability distribution over bi-infinite configurations. Specifically, a *simulator* S of process P is an object where, given an instance of a semi-infinite “past” $x_{:0} = \dots, x_{-3}, x_{-2}, x_{-1}$, a query of S yields a sample of the “future” $X_{0:} = x_0, x_1, \dots$ from the conditional probability distribution $\Pr(X_{0:} | X_{:0} = x_{:0})$. See Fig. 1.

Physical systems, under certain assumptions such as thermal equilibrium, manifest stationary stochastic processes. When we refer to the simulation of a physical system, what we mean is the simulation of these processes.

How are these simulators implemented? Two common formalisms are *Markov Chains* [14, 15] and *Hidden Markov Models* (HMM) [10, 16, 17]. The latter can be significantly more compact in their representation³ and,

² Throughout, T denotes the effective temperature $k_B T$.

³ For example, the Simple Nonunifilar Source (SNS) process requires an infinite-state Markov chain and consequently requires infinite memory for simulation, while its HMM requires only two states and so a single bit of memory [18].

for this reason, are sometimes the preferred implementation choice. Here, we employ a particularly useful form of HMM generators.

These HMMs represent the generating mechanism for a given process by a tuple $\{\mathcal{S}, \mathcal{A}, \{T^{(x)} : x \in \mathcal{A}\}, \}$ where \mathcal{S} is a finite set of states called *causal states*, \mathcal{A} is a finite discrete alphabet and $\{T^{(x)} : x \in \mathcal{A}\}$ are $|\mathcal{S}| \times |\mathcal{S}|$ substochastic symbol-labeled transition matrices. The latter's sum $\mathbf{T} = \sum_{x \in \mathcal{A}} T^{(x)}$ is a stochastic matrix. A *unifilar* HMM is one in which each row of each substochastic matrix has at most one nonzero element.⁴

ϵ -Machine A given stochastic process can be correctly generated by any number of unifilar HMMs. The one requiring the minimum amount of memory for implementation is called the ϵ -machine [23] and was first introduced in Ref. [24]. A process' *statistical complexity* C_μ [23] is the the Shannon entropy of the ϵ -machine's stationary state distribution: $C_\mu = H(\mathcal{S}) = -\sum_{\sigma \in \mathcal{S}} \Pr(\sigma) \log_2 \Pr(\sigma)$. Key to our analysis of classical simulator resources, it measures the minimal memory for a unifilar simulator of a process. C_μ has been determined for a wide range of physical systems [25–31]. Helpfully, it and companion measures are directly calculable from the ϵ -machine, many in closed-form [32].

Ising ϵ -machine How do we construct the ϵ -machine that simulates the process $P(N, T)$?

First, we must define process' *Markov order* [15]: the minimum history length R required by any simulator to correctly continue a configuration.⁵ Specifically, R is the smallest integer such that:

$$\mathbb{P}(X_t | \dots, X_{t-2}, X_{t-1}) = \mathbb{P}(X_t | X_{t-R}, \dots, X_{t-2}, X_{t-1}) .$$

Reference [13, Eqs. (84) – (91)] shows that $P(N, T)$ has Markov order N for any finite and nonzero temperature T . One concludes that sufficient information for continued generation is contained in the configuration of the N previously generated spins. More importantly, the ϵ -machine that simulates $P(N, T)$ has 2^N states and those states are in one-to-one correspondence with the set of length- N spin configurations.

Second, another key process characteristic is its *cryptic order* [33, 34]: the smallest integer K such that $H(\mathcal{S}_K | X_0, X_1 \dots) = 0$, where $H[W|Z]$ is the conditional entropy [35] and \mathcal{S}_K is the random variable for the

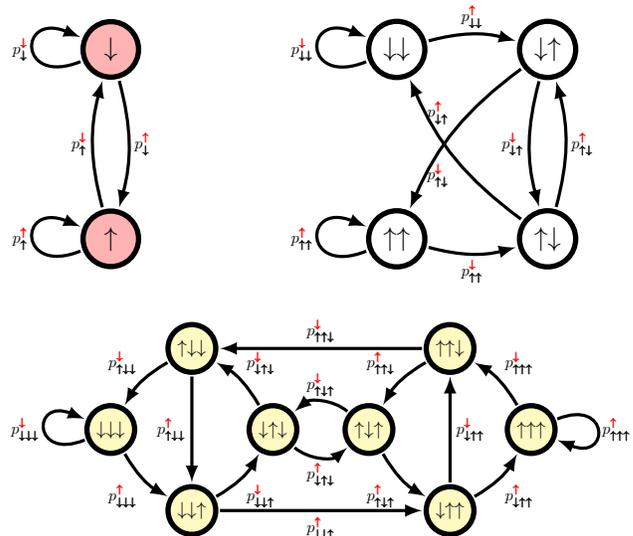


FIG. 2. ϵ -Machine generators of 1D-configuration stochastic processes in Dyson-Ising systems of increasing correlational complexity ($N = 1, 2, 3$): $P(1, T)$ (top left), $P(2, T)$ (top right) and $P(3, T)$ (bottom).

K^{th} state of the ϵ -machine after it generated symbols $X_0, X_1 \dots$. Using the fact that ϵ -machine states are in one-to-one correspondence with the set of length- N spin configurations, it is easy to see that $P(N, T)$'s cryptic order $K = N$.

Figure 2 shows the unifilar HMM generators (ϵ -machines) of the processes $P(N, T)$ for $N = 1, 2$, and 3. Let's explain.

Consider the spin process $P(1, T)$, a Markov-order $R = 1$ process. To generate the process we only need to remember the last spin generated. The ϵ -machine (Fig. 2 top-left) has two states, \uparrow and \downarrow . If the last observed spin is \uparrow , then the current state is \uparrow and if it is \downarrow , the current state is \downarrow . We denote the probability of a \downarrow spin given a previous \uparrow spin by $p_{\uparrow\downarrow}$. The probability of an \uparrow spin following a \uparrow spin is the complement.

Consider the process $P(2, T)$ with Markov-order $R = 2$ and so longer-range interactions. Sufficient information for generation is contained in the configuration of the two previously generated spins. Thus, the ϵ -machine (Fig. 2 top-right) has four states that we naturally label $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, and $\downarrow\downarrow$. If the last observed spin pair $x_{-1}x_0$ is $\uparrow\downarrow$, the current state is $\uparrow\downarrow$. Given this state, the next spin will be \uparrow with probability $p_{\uparrow\downarrow\uparrow}$ and \downarrow with probability $p_{\uparrow\downarrow\downarrow}$. Note that this scheme implies that each state has exactly two outgoing transitions. That is, not all transitions are allowed in the ϵ -machine.

Having identified the state space, we may calculate the ϵ -machine transition probabilities $\{T^{(x)}\}_{x \in \mathcal{A}}$. We first compute the transfer matrix \mathbf{T} [36] and then extract con-

⁴ A fledgling literature on minimal *nonunifilar* HMMs [19] exists, but constructive methods are largely lacking and, as a consequence, much less is known [20–22].

⁵ More precisely, we mean that an ensemble of simulators must be able to yield an ensemble of configurations that agree (conditioned on that past) with the process' configuration distribution.

ditional probabilities, following Ref. [13]. (See App. B for details.) The statistical complexity C_μ follows straightforwardly from the ϵ -machine.

q-Machine By studying a specific process (similar to the ϵ -machine in top-left of Fig. 2), Ref. [37] recently demonstrated that quantum mechanics can simulate stochastic processes using memory capacity less than C_μ . This motivates a search for more efficient quantum simulators of other stochastic processes.

A class of such simulators, called *q-machines*, applicable to arbitrary processes, was introduced in Ref. [6]. This construction depends on an encoding length L , each with its own q-machine and its quantum cost $C_q(L)$. The cost $C_q(L)$ saturates at a particular length, which was shown to be the process' cryptic order K , introduced above [34]. And so, we restrict ourselves to this choice ($L = K$) of encoding length and refer simply to the q-machine and its cost C_q .

The q-machine's quantum memory C_q is upper-bounded by C_μ , with equality only for the special class of zero-cryptic-order processes [34]. And so, C_μ/C_q gives us our quantitative measure of *quantum advantage*. Efficient methods for calculating C_q were introduced by Ref. [38] using spectral decomposition. Those results strongly suggest that the q-machine is the most memory-efficient among all unifilar quantum simulators, but as yet there is no proof.⁶ The quantum advantage C_μ/C_q has been investigated both analytically [6, 38, 41–43] and experimentally [44].

The q-machine is straightforward to construct from a given ϵ -machine. It consists of a set $\{|\eta_i\rangle\}$ of pure quantum *signal states* in one-to-one correspondence with the classical causal states $\sigma_i \in \mathcal{S}$. Each signal state $|\eta_i\rangle$ encodes the set of length- K (cryptic order) sequences that may follow σ_i , as well as each corresponding conditional probability:

$$|\eta_i\rangle \equiv \sum_{w \in |\mathcal{A}|^K} \sum_{\sigma_j \in \mathcal{S}} \sqrt{\mathbb{P}(w, \sigma_j | \sigma_i)} |w\rangle |\sigma_j\rangle,$$

where w denotes a length- K sequence and $\mathbb{P}(w, \sigma_j | \sigma_i) = \mathbb{P}(X_0 \cdots X_{K-1} = w, \mathcal{S}_{K-1} = \sigma_j | \mathcal{S}_0 = \sigma_i)$. The resulting Hilbert space is the product $\mathcal{H}_w \otimes \mathcal{H}_\sigma$. Factor space \mathcal{H}_σ is of size $|\mathcal{S}|$, the number of classical causal states, with basis elements $|\sigma_i\rangle$. Factor space \mathcal{H}_w is of size $|\mathcal{A}|^K$, the number of length- K sequences, with basis elements $|w\rangle = |x_0\rangle \cdots |x_{K-1}\rangle$. For $P(N, T)$'s ϵ -machines, $|\mathcal{S}| = 2^N$ and $|\mathcal{A}|^K = 2^N$. The q-machine's density matrix ρ is defined

⁶ As in the classical case, nonunifilar quantum simulators are much less well understood [22, 39, 40].

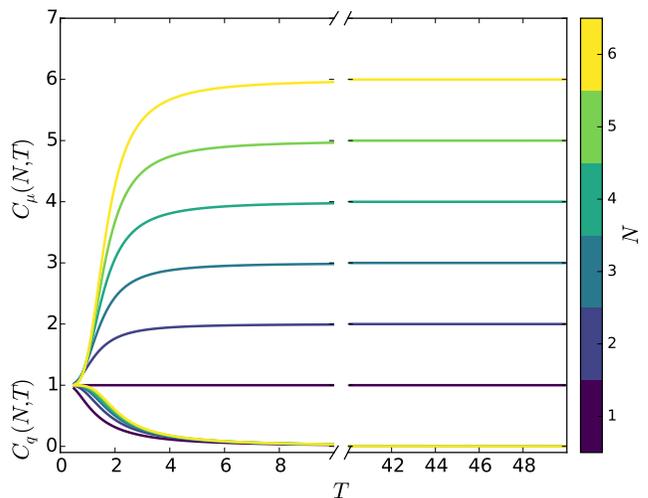


FIG. 3. Classical memory $C_\mu(N, T)$ and quantum memory $C_q(N, T)$ required for simulating the Dyson-Ising processes $P(N, T)$ for $N = 1, \dots, 6$, a range of temperatures $T = 1, \dots, 50$ and $\delta = 2$. Note the dramatic difference in behaviors. $C_\mu(\cdot)$ and $C_q(\cdot)$ both are increasing functions of N . However, $C_\mu(\cdot)$ is an increasing function of T and $C_q(\cdot)$ is decreasing function of T and bounded by 1, independent of N and T .

by:

$$\rho = \sum_i \pi_i |\eta_i\rangle \langle \eta_i|, \quad (1)$$

where $\{\pi_i\}$ is the stationary distribution over the ϵ -machine's states. This is determined from the left eigenvector of transfer matrix \mathbf{T} corresponding to the eigenvalue 1. From the density matrix ρ , it is straightforward to calculate $C_q = S(\rho)$ — ρ 's von Neumann entropy.

V. ANALYSIS

We begin by considering the case where couplings decay with exponent $\delta = 2$. Figure 3 displays $C_\mu(N, T)$ and $C_q(N, T)$ —the C_μ and C_q of processes $P(N, T)$ —versus T for $N = 1, \dots, 6$. The most striking feature is that the classical and quantum memory requirements exhibit qualitatively very different behaviors.

Classical memory increases with T , saturating at $C_\mu = N$, since all transitions become equally likely at high temperature. As a result there are 2^N equally probable causal states and this means one needs N bits of memory to store the system's current state. For example, in the nearest-neighbor Ising model (process $P(1, T)$) high temperature makes spin- \uparrow and spin- \downarrow , and thus the cor-

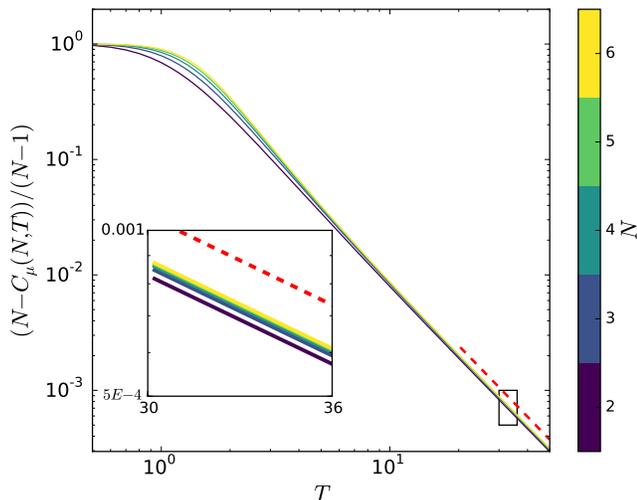


FIG. 4. Rescaling the classical memory requirement $C_\mu(N, T)$ to $(N - C_\mu)/(N - 1)$ shows a tight data collapse, which is especially strong at high temperatures ($T > 2$). The asymptotic behavior is a power-law with scaling exponent $\gamma = 2$. The inset zooms in to show C_μ 's convergence with increasing N . While the figure shows the case $\delta = 2$, but the slope γ at high T is independent of T .

responding states, equally likely.⁷

Also, in the low-temperature limit, this system is known to yield one of only two equally likely configurations—all spin- \uparrow or all spin- \downarrow . In other words, at low temperature p_\uparrow^\downarrow and p_\downarrow^\downarrow converge to zero, while p_\uparrow^\uparrow and p_\downarrow^\uparrow converge to one.⁸ This is reflected in the convergence of all curves at $C_\mu = 1$. Equivalently, this means one needs only one bit of memory to store the current state.

We can similarly understand the qualitative behavior of $C_q(N, T)$ for a fixed N . As temperature increases, all length- N signal states become equivalent. This is the same as saying that all length- N spin configurations become equally likely. As a consequence, the signal states approach one another and, thus, $C_q(N, T)$ converges to zero.

In the low-temperature limit, the two N - \uparrow and N - \downarrow configurations are distinguished by the high likelihood of neighboring spins being of like type. This leads to a von Neumann entropy of $S(\rho) = 1$.

Figure 3 reveals strong similarities in the form of $C_\mu(T)$ at different N . A simple linear scaling leads to a sub-

⁷ At $T = \infty$ these processes have only a single causal state and thus $C_\mu = 0$. This is a well known discontinuity that derives from the sudden predictive-equivalence of all of the causal states there.

⁸ It should be pointed out that at any finite temperature p_\uparrow^\downarrow and p_\downarrow^\uparrow are nonzero and, therefore, the ϵ -machine states remain strongly-connected.

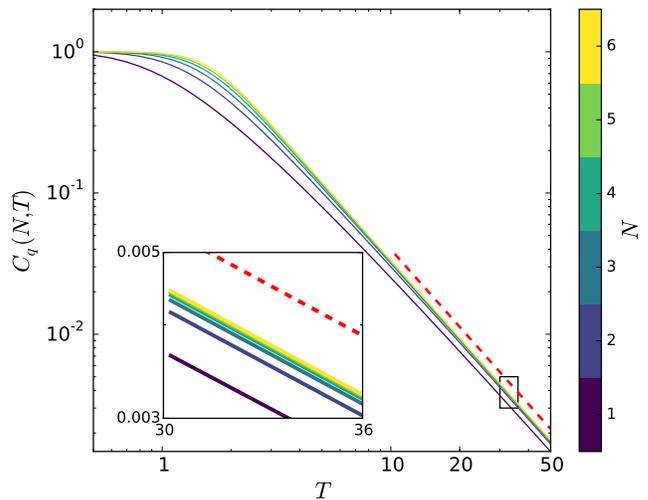


FIG. 5. Quantum memory $C_q(N, T)$ too shows a data collapse in N that is also especially tight at high temperature ($T > 2$). The asymptotic behavior is a power-law with numerically estimated scaling exponent $\alpha = 2$. The lower inset zooms in to highlight convergence with increasing N . While the figure shows the case with $\delta = 2$, but the slope α at high T is independent of T .

stantial data collapse, shown in Fig. 4. The scaled curves $(N - C_\mu)/(N - 1)$ exhibit power-law behavior in T for $T > 2$. Increasing the temperature to $T = 300$ (beyond the scale of the Fig. 4) shows that this scaling is $\gamma \simeq 2.000$. The scaling indicates how the classical memory saturates at high temperature.

This behavior is generic for different coupling decay values $\delta > 1$ and, more to the point, the scaling (γ) is independent of δ . We do not consider $\delta < 1$, where the system energy becomes nonextensive.

Now, we can analyze the decrease in C_q with temperature. Figure 5 shows that C_q is also a power-law in T . By measuring this scaling exponent in the same way as above, we determine $\alpha = 2.000$. Furthermore, analysis shows (App. D) that:

$$C_q(N, T) \propto \frac{T^2}{\log_2(T)}.$$

This verifies and adds detail to our numerical estimate. This behavior is generic for different coupling decay values $\delta > 1$ and, moreover, the scaling exponent α is independent of δ . It is interesting to note that in this case no data-collapse rescaling is required. The exponent α directly captures the extreme compactness of high temperature quantum simulations.

Taking these results together, we can now appreciate the substantial relative advantage of quantum versus classical simulators.

Define the *quantum advantage* η as the ratio of the minimum required memory for the classical simulation to that for the quantum simulation:

$$\eta(N, T) \equiv C_\mu(N, T)/C_q(N, T).$$

For fixed temperature $T \gtrsim 2$, $C_\mu(N, T)$ is approximately linear in N and for a fixed N it is approximately independent of T . As a consequence, the asymptotic quantum advantage scales as:

$$\eta(N, T) \propto N \frac{T^2}{\log_2(T)},$$

which increases faster than any T^r for $r < 2$. Thus, the answer to our motivating question is that the quantum advantage does in fact display a scaling behavior, increasing without bound with interaction range N and even faster with temperature T .

VI. CONCLUSION

It is notoriously hard to find quantum advantage and even harder to prove [45]. Here, we found just such an advantage in the realm of stochastic-process simulation. Concretely, we analyzed the N -nearest neighbor Ising spin system and demonstrated that its quantum advantage displays generic scaling behavior—quadratic in temperature and linear in interaction range. What does this mean? The most striking conclusion is that a

highly interacting classical system can be simulated with unbounded quantum advantage. Given the simplicity of the Ising system, we conjecture that this scaling behavior may be a universal feature of quantum advantage in the simulation of extended physical systems.

The Ising model has contributed great insights to condensed matter physics, however, it is classical. Given our examining the difference between classical and quantum simulators, it is natural to wonder about this difference in the context of a truly quantum Hamiltonian. Is the quantum advantage amplified? Are there systems for which we find no quantum advantage? What is the distinguishing characteristic?

For finite-range interaction in one dimension, there is no Ising phase transition. How might the quantum advantage change in the presence of such a transition? Given that the quantum advantage here scaled with the interaction range, we might expect that near the critical temperature, where long-range interactions are important, the quantum advantage is amplified.

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Appendix A: Why the Dyson model?

The ferromagnetic Ising spin linear chain with finite-range interaction cannot undergo a phase transition at any positive temperature [46]. In contrast, the Dyson model has a standard second-order phase transition for a range of δ . It was analytically proven by Dyson [9] that a phase transition exists for $1 < \delta < 2$. The existence of a transition at $\delta = 2$ was proven much later on [47]. It is also known that there exists no phase transition for $\delta > 3$ [48] where it behaves as a short-range system. Finally, it was demonstrated numerically that the parameter regime $2 < \delta \leq 3$ contains a phase transition [49], however, this fact has resisted analytical proof. For $\delta \leq 1$, the model is considered nonphysical since the energy becomes non-extensive.

For these reasons we selected the Dyson spin model as it provides the simplicity of 1D configurations, while generating nontrivially correlation spin configurations.

Appendix B: ϵ -Machine Construction

We show how to construct the ϵ -machine simulator of the process $P(N, T)$. Consider a block of spins of length $2N$, divided equally into two blocks. We denote spins in the left (L) and right (R) halves by: s_i^L and s_i^R for $i = 1, \dots, N$, respectively. We map the left and right block configurations each to an integer η_X by:

$$\eta_* = \sum_{i=1}^N \left(\frac{s_i^* + 1}{2} \right) 2^{i-1},$$

$* \in \{L, R\}$. The blocks internal energies are given by:

$$X_{\eta_*} = -B \sum_{i=1}^N s_i^* - \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} J_i s_k^* s_{k+i}^*,$$

and the correlated energy between two blocks is:

$$Y_{\eta_L, \eta_R} = - \sum_{i=1}^N \sum_{k=1}^i J_i s_{N-k+1}^L s_k^R.$$

With these we construct the transfer matrix:

$$V_{\eta_L, \eta_R} = \exp \left(-\frac{1}{T} (1/2 X_{\eta_L} + Y_{\eta_L, \eta_R} + 1/2 X_{\eta_R}) \right).$$

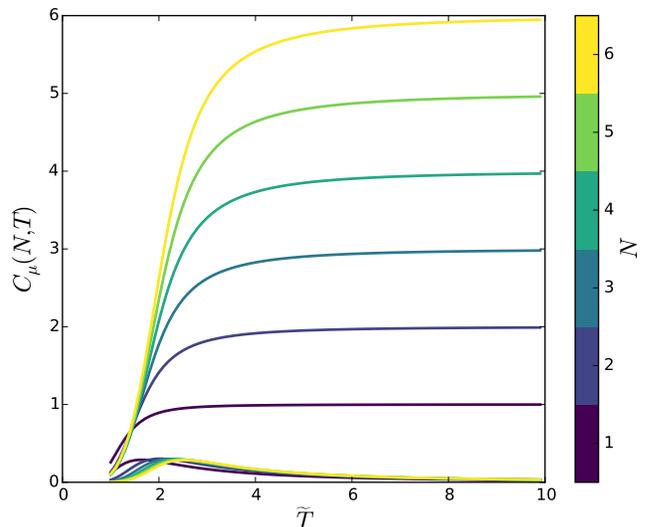


FIG. 6. Magnetic field effects on classical $C_\mu(N, T)$ and quantum memory $C_q(N, T)$ requirements for simulating the process generated by Hamiltonian $\hat{\mathcal{H}}_N$ for $N = 1, \dots, 6$ and a range of temperatures $T = 1, \dots, 50$ at $B = 0.3$.

Reference [13] shows that the ϵ -machine labeled-transition matrices can be written as:

$$T_{\eta_0, \eta_1}^{(x)} = \begin{cases} \frac{1}{\lambda} V_{\eta_0, \eta_1} \frac{u_{\eta_1}}{u_{\eta_0}} & \eta_1 = (\lfloor \frac{\eta_0}{2} \rfloor + x * 2^{N-1}) \\ 0 & \eta_1 \neq (\lfloor \frac{\eta_0}{2} \rfloor + x * 2^{N-1}) \end{cases}.$$

Then the ϵ -machine simulator of $P(N, T)$ is $\{\mathcal{S}, \mathcal{A}, \{T^{(x)}\}_{x \in \mathcal{A}}\}$ where $\mathcal{A} = \{0, 1\}$ and $\mathcal{S} = \{i : 1 \leq i \leq 2^N\}$.

Appendix C: Presence of Magnetic Field

Naturally, one might ask how our results are modified by the presence of an external magnetic field. Consider the one-dimensional ferromagnetic Ising spin chain with Hamiltonian:

$$\hat{\mathcal{H}}_N = - \sum_i \sum_{k=1}^N \frac{J_0}{k^\delta} s_i s_{i+k} - \sum_i B s_i.$$

Figure 6 shows that, due to the symmetry breaking at low temperature, both $C_q(N, T)$ and $C_\mu(N, T)$ converge to zero. (All the spins at low temperature align with magnetic field and, as a consequence, no memory is needed.) The high temperature behaviors for both functions are the same as before and the quantum advantage remains the same.

Appendix D: High Temperature Behavior

Consider first the case $N = 1$. Due to symmetry we have $p \equiv \Pr(\uparrow | \uparrow) = \Pr(\downarrow | \downarrow) = N/D$, where $N = \exp(\beta J)$ and $D = \exp(\beta J) + \sqrt{\exp(-2\beta J)}$ with $\beta = 1/T$. At high temperature β is small and we have:

$$\begin{aligned} D &= 2 + \beta^2, \\ N &= 1 + \beta + \beta^2. \end{aligned}$$

Again, due to symmetry we have $\pi_1 = \pi_2 = 1/2$. Therefore, the density matrix in Eq. (1) is:

$$\rho = \begin{pmatrix} 1/2 & \sqrt{p(1-p)} \\ \sqrt{p(1-p)} & 1/2 \end{pmatrix},$$

which has two eigenvalues $\beta^2/4$ and $1 - \beta^2/4$. As a con-

sequence, C_q — ρ 's von Neumann entropy—is:

$$\begin{aligned} C_q &= S(\rho) \\ &\simeq - \left(\frac{\beta^2}{4} \log_2 \frac{\beta^2}{4} + \left(1 - \frac{\beta^2}{4}\right) \log_2 \left(1 - \frac{\beta^2}{4}\right) \right) \\ &\simeq \frac{\log_2(T)}{2T^2}. \end{aligned}$$

Examining the numerator, for any $r > 0$ we have $\log_2(T) < T^r$. So, for large T and for all $r > 0$:

$$\frac{1}{T^2} < \frac{\log_2(T)}{T^2} < \frac{1}{T^{r+2}}.$$

This explains the fat tails of C_q for large T . More to the point, it shows that for $N = 1$ the scaling exponent is $\alpha = 2$.

Increasing the temperature, the interaction between spins weakens. At high temperature the only important neighbor is the nearest neighbor. And so, the high-temperature behavior is similar to the case of $N = 1$ and is independent of N .