Rational Insurance with Linear Utility and Perfect Information

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Abstract

We present a mathematical solution to the insurance puzzle. Our solution only uses time-average growth rates and makes no reference to risk preferences. The insurance puzzle is this: according to the expectation value of wealth, buying insurance is only rational at a price that makes it irrational to sell insurance. There is no price that is beneficial to both the buyer and the seller of an insurance contract. The puzzle why insurance contracts exist is traditionally resolved by appealing to utility theory, asymmetric information, or a mix of both. Here we note that the expectation value is the wrong starting point – a legacy from the early days of probability theory. It is the wrong starting point because not even the most basic models of wealth (random walks) are stationary, and what the individual experiences over time is not the expectation value. We use the standard model of noisy exponential growth and compute time-average growth rates instead of expectation values of wealth. In this new paradigm insurance contracts exist that are beneficial for both parties.

Rational insurance with linear utility and perfect information

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1 Summary: The insurance puzzle and two treatments

We revisit the question why insurance contracts exist, unresolved in the original "expected-wealth paradigm" of 17th-century probability theory. Bernoulli (1738) treated this question by introducing the "utility paradigm" in the 18th century. We propose an alternative answer, using what we call the "time paradigm" that makes use of more recent developments in the understanding of stochastic processes.

The issues at hand are ostensibly about insurance, interesting from the perspectives of both actuarial science and decision theory. Through decision theory their relevance extends to much of formal economics. For instance, the trivial extension from insurance contracts to derivatives such as options, futures, and credit default swaps, demonstrates the significance of the time paradigm for modern financial markets.

1.1 The insurance problem of the expected-wealth paradigm

Why shouldn't insurance contracts exist? The expected-wealth paradigm is the model of human behaviour that posits that humans act so as to maximise the

expectation value of their wealths. It predicts the absence of insurance contracts as follows:

1. To be viable, an insurer has to charge an insurance premium of at least the expectation value of any claims that may be made against it, called the "net premium," (Kaas *et al.*, 2008, p. 1).

2. The insurance buyer therefore has to be willing to pay more than the net premium so that an insurance contract may be successfully signed.

3. Under the expected-wealth paradigm it is irrational to pay more than the net premium, and therefore insurance contracts should not exist.

An insurance contract can only ever be beneficial to one party – it has the anti-symmetric property that the expectation value of the gain of one party is the expectation value of the loss of the other party.

1.2 Treatment 1: the utility paradigm

Bernoulli (1738) identified the rationality model in step 3 above as problematic and devised the utility paradigm. He observed that money may not translate linearly into usefulness and assigned to any individual an idiosyncratic utility function $U(W)$ that maps wealth W into usefulness U, which he posited as the true quantity whose expected change is maximised in a decision. Since the utility function of the insurance buyer may be non-linear, it is possible that the buyer experiences a positive change in the expectation value of his utility from signing an insurance contract, even if the expectation value of his monetary wealth decreases. Therefore, the utility paradigm does not rule out the existence of insurance contracts. It does not share the anti-symmetric property of the expected-wealth paradigm. The expectation value of the gain of utility of one party need not be the expectation value of the loss of utility of the other party.

In fact, since different people have different wealths or different utility functions or both, the change in the expectation value of utility experienced by one party is almost unrestricted by the change experienced by the other party. This is problematic and led Kelly Jr. (1956, p. 918) to comment that the utility paradigm is "too general to shed any light on the specific problems". The paradigm affords too much freedom by appealing to individual differences and by allowing too broad a class of utility functions. As in any paradigm, generality comes at the expense of predictive power, and in the case of the utility paradigm, the predictive power is often limited to the point of practical uselessness. Utility theory dominates formal economics but plays no major role in practical actuarial work[1].

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[1]Limiting the likelihood of default is a more powerful idea in the field, for instance to set insurance fees.

1.3   Treatment 2: the time paradigm

In agreement with Bernoulli we take issue with step 3 of Section 1.1. It does indeed imply an inappropriate model of rationality. But the correction we propose is entirely different from Bernoulli's in its epistemology. Step 3 relies on the assumption that people optimise the rate of change of the expectation value of wealth. Before the development of statistical mechanics in the 19th century there was little reason to doubt the wisdom of this assumption. This is so because no one had questioned whether the change of the expectation value of an observable in a stochastic process is the same as the time-average of the change of the observable. This question was only posed in the late 19th century: is taking expectation values equivalent to averaging over time? We now know that this equivalence only holds under special conditions, the study of which is known as ergodic theory. Growth processes violate these conditions. Statistical mechanics also stresses the physical meaning of an expectation value. This is an average over an ensemble of infinitely many statistically identical systems, often called "parallel universes."

The "time paradigm" recognises all this. The time average growth rate under reasonable dynamics (the standard is multiplicative dynamics) behaves very differently from the rate of change of the expectation value. Typically there exists a range of prices where time average growth rates increase for both parties when an insurance contract is signed.

Without appealing to utility functions, the time paradigm predicts the existence of insurance contracts with deep conceptual consequences. Any variable representing a growing quantity is a far-from-equilibrium process. Insofar as economic models represent (positively or negatively) growing entities, equilibrium intuition is misleading here. Acknowledging this, a business deal is seen to happen because it benefits both parties, whereas equilibrium intuition suggests that there will generally be one winner and one loser.

2   The model – a shipping example

We use the most basic model of an insurance contract, considered *e.g.* by Rothschild & Stiglitz (1976). The model is chosen because it is simple and has all the necessary elements to demonstrate the failure of the expected-wealth paradigm and the different fixes under the utility paradigm and under the time paradigm. Bernoulli (1738) applied the model in the context of shipping, and we will do the same for illustration[2]. To make the model more concrete and demonstrate that it works in a practically interesting regime, we use example values for its parameters.

A St. Petersburg merchant sends a ship to Amsterdam. Upon the safe arrival of the ship the merchant will make a gain of $G = \$4,000$. However, with probability $p = 0.05$ the ship, whose replacement cost is $C = \$30,000$, will be

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[2] The main focus of (Bernoulli, 1738), the St. Petersburg paradox, was resolved using the time paradigm in (Peters, 2011*a*).

lost. The journey is known to take $\Delta t = 1$ month. The ship owner's wealth when the ship sets sail at time t is $W_{\mathrm{own}} = \$100,000$. An insurer proposes a contract stipulating that, should the ship be lost, the owner shall receive the replacement cost of the ship and his lost profit, $L = G + C$, for an insurance fee of $F = \$1,800$. If the two parties sign the contract the owner will receive $G - F$ with certainty after one month, whereas the insurer will receive F with probability $1 - p$ and lose $L - F$ with probability p. The contract is completely specified by the insured loss L and the fee F (and the payout condition, of course). We assume that both parties have perfect information, *i.e.* both know the (correct) probability of the ship being lost, and there is no possibility of fraud or default. Should the owner sign the insurance contract, and did the insurer act in his own interest by proposing it?

3 The expected-wealth paradigm and its problems

The expected-wealth paradigm, in the language of economics, assumes that humans are risk neutral, *i.e.* they have no preference between gambles whose expected changes in wealth are identical (over a given time interval). This assumption has been known to be flawed at least since 1713 (Montmort, 1713, p. 402). We discuss it here to better understand the origin of the utility paradigm and enable a conceptually different approach. Under the expected-wealth paradigm humans act to maximise the rate of change of the expectation value of their wealth,

$$\langle r \rangle = \frac{\langle \Delta W \rangle}{\Delta t} = \frac{\langle W(t + \Delta t) \rangle - W(t)}{\Delta t}. \tag{1}$$

The attractiveness of an insurance contract is then judged by computing the change in $\langle r \rangle$ that results from signing the contract.

The owner's perspective

Without insurance, the rate of change of the expectation value of the owner's wealth is

$$\langle r \rangle_{\mathrm{own}}^{\mathrm{un}} = \frac{(1 - p)G - pC}{\Delta t} = \frac{G - pL}{\Delta t}, \tag{2}$$

or $\$2,300$ per month using the example parameters.

With insurance the (certain) rate of change of wealth is

$$\langle r \rangle_{\mathrm{own}}^{\mathrm{in}} = \frac{G - F}{\Delta t}. \tag{3}$$

In the example this is $\$2,200$ per month.

The change in the rate of change of the expectation value of the owner's wealth resulting from entering into the insurance contract is

$$\delta \langle r \rangle_{\mathrm{own}} = \langle r \rangle_{\mathrm{own}}^{\mathrm{in}} - \langle r \rangle_{\mathrm{own}}^{\mathrm{un}} = \frac{pL - F}{\Delta t}, \tag{4}$$

here $-\$100$ per month. This change is marked by δ rather than Δ to clarify that it is not the change over Δt of a time-varying quantity.

Buying insurance in this example reduces the rate of change of the expectation value of the owner's wealth. Hence, the owner should not sign the contract according to the expected-wealth paradigm.

The insurer's perspective
Without insurance, the (certain) rate of change in wealth of the insurer is

$$\langle r \rangle_{\text{ins}}^{\text{un}} = 0 \tag{5}$$

(the insurer does no business).

With insurance, the rate of change of the expectation value of the insurer's wealth is

$$\langle r \rangle_{\text{ins}}^{\text{in}} = \frac{F - pL}{\Delta t}, \tag{6}$$

or $+\$100$ per month in the example.

The change in the rate of change of the expectation value of the insurer's wealth resulting from entering into the contract is

$$\delta \langle r \rangle_{\text{ins}} = \langle r \rangle_{\text{ins}}^{\text{in}} - \langle r \rangle_{\text{ins}}^{\text{un}} = \frac{F - pL}{\Delta t}, \tag{7}$$

here $+\$100$ per month. Hence the insurer should sign the contract according to the expected-wealth paradigm. But since the owner should not sign, no deal will be made.

Crucially, irrespective of the parameter values, what the insurer gains is always precisely what the owner loses (*c.f.* (Eq. 4) and (Eq. 7)),

$$\delta \langle r \rangle_{\text{ins}} = -\delta \langle r \rangle_{\text{own}}. \tag{8}$$

This anti-symmetry makes insurance a fundamentally unsavoury business – a zero-sum game where one party wins at the expense of the other. The existence of such contracts in the real world requires asymmetries between the contracting parties. Buyer and seller may have different access to information. They may make different subjective assessments of the risk being insured (embodied in their estimates of model parameters such as p and L). Or they may simply deceive, coerce, or gull the other party into the necessary sub-optimal decision.

Problem with the expected-wealth paradigm:
No price exists that makes an insurance contract beneficial to both parties, and yet insurance contracts exist.

We summarise the two perspectives in Table 1, and illustrate the effect in Fig. 1.

Table 1: Changes in the rates of change of the expectation value of wealth for owner and insurer. None of the expressions contains the wealth of either party.

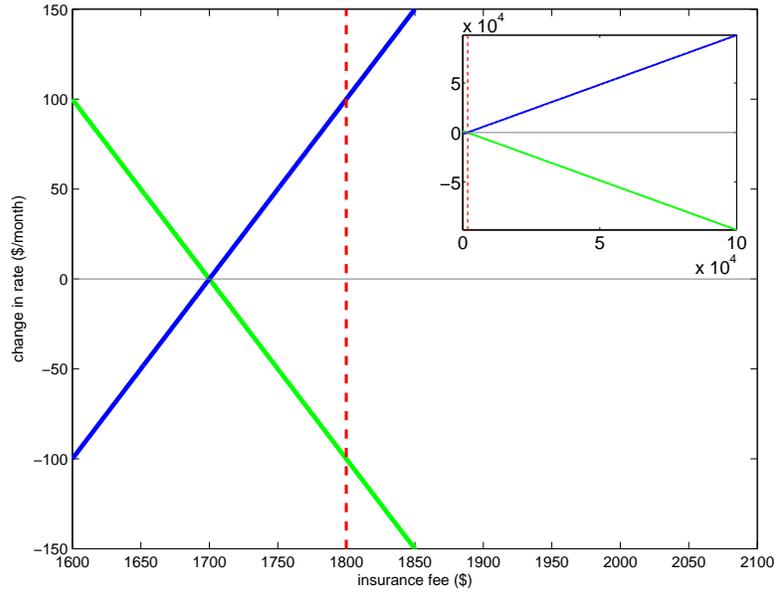| $\langle r \rangle$	Owner	Insurer
insured	$\langle r \rangle_{\text{own}}^{\text{in}} = \frac{G-F}{\Delta t}$	$\langle r \rangle_{\text{ins}}^{\text{in}} = \frac{F-pL}{\Delta t}$
uninsured	$\langle r \rangle_{\text{own}}^{\text{un}} = \frac{(1-p)G-pC}{\Delta t}$	$\langle r \rangle_{\text{ins}}^{\text{un}} = 0$
difference	$\delta \langle r \rangle_{\text{own}} = \frac{pL-F}{\Delta t}$	$\delta \langle r \rangle_{\text{ins}} = -\delta \langle r \rangle_{\text{own}} = \frac{F-pL}{\Delta t}$

Figure 1: Change in the rate of change of the expectation value of wealth resulting from signing an insurance contract versus insurance fee. Green line: owner. Blue line: insurer. The straight lines of opposite slopes cross at zero, meaning that there is no price where both lines lie above the zero line – the winning party only gains at the expense of the losing party. The vertical dashed line indicates the proposed insurance fee, which appears attractive to the insurer and unattractive to the owner. Inset: horizontal range extended to $100,000.

4 Solution in the utility paradigm

The model of human decision making of the expected-wealth paradigm predicts that insurance contracts do not exist. This is in disagreement with observations. D. Bernoulli (1738) concluded that, apparently, humans use a different criterion when deciding whether to sign insurance contracts. He introduced a non-linear mapping of money, declaring: if humans don't act to maximise the expectation value of money, then let's say they act to maximise the expectation value of some sub-linearly increasing function of money. He called this function the "utility function" and left it largely unspecified, using the logarithm in computations and pointing out that it produces essentially the same results as a square root function[3]. A dollar to a rich man has a value different from a dollar to a poor man; some people like gambling, others are more prudent.

In other words, the idiosyncrasy of human beings was invoked. This does not so much resolve the puzzle as provide a description in mathematical terms that is consistent with a description in everyday language: the expected-wealth paradigm does not work. The introduction of a non-linear utility function breaks the equality of the locations of zero-crossings of the value of the insurance contract for both parties. It is this equality that makes insurance impossible in the expected-wealth paradigm. Any perturbation that breaks it and creates a win-win range of fees suffices to solve the mathematical problem. Introducing largely arbitrary non-linear utility functions, possibly different functions for owner and insurer constitutes such a perturbation. An example is shown in Fig. 2, where utility is given by the square-root of wealth, $U(W) = \sqrt{W}$ (the first ever utility function to be suggested, by Cramer in a 1728 letter cited by Bernoulli (1738)). A more satisfying resolution of the puzzle would explain the rationale behind such a perturbation – why a square-root? Why a logarithm? Why re-value money at all?

In general we consider the rate of change of the expectation value of utility,

$$\langle r_u \rangle = \frac{\langle \Delta U \rangle}{\Delta t} = \frac{\langle U(t + \Delta t) \rangle - U(t)}{\Delta t}. \tag{9}$$

Again using the shipping example, the equations corresponding to Section 3 are:

The owner's perspective

Without insurance, the rate of change of the expectation value of the owner's utility is

$$\langle r_u \rangle^{\text{un}}_{\text{own}} = \frac{(1-p)U_{\text{own}}(W_{\text{own}} + G) + pU_{\text{own}}(W_{\text{own}} - C) - U(W_{\text{own}})}{\Delta t}. \tag{10}$$

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[3] Bernoulli (1738) is credited with computing changes in the expectation values of utility functions although a careful reading of his work shows that he computed something slightly different (Peters, 2011a). The difference was small in his context, and later researchers assumed he had meant to compute changes in expectation values (Laplace, 1814, p. 439–442).
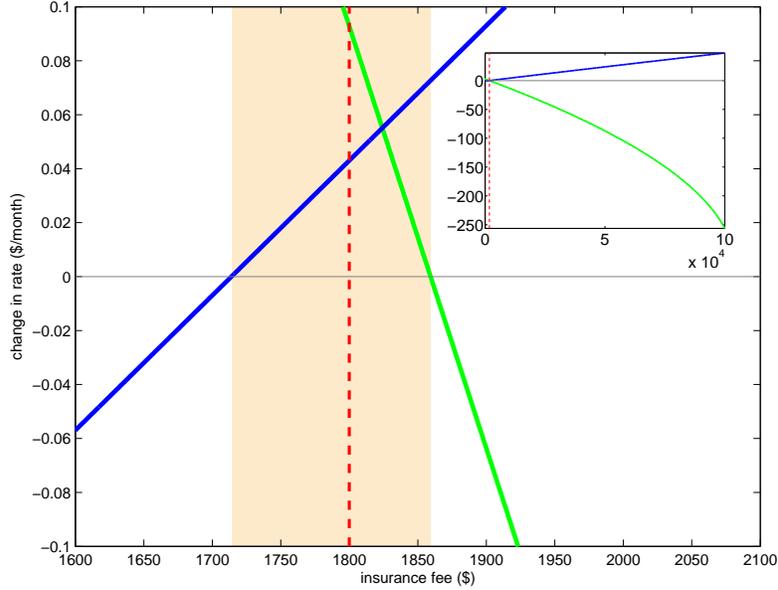
Figure 2: Change in the rate of change of the expectation value of square-root utility resulting from signing an insurance contract versus insurance fee. Green line: owner. Blue line: insurer. The vertical dashed line indicates the proposed insurance fee. Shaded region: Introducing the non-linear utility function is a sufficient perturbation to create a regime where both parties gain. Both the green and blue lines are above the grey zero line here. However, the non-linear function is ill-constrained by the formalism, which limits predictive power. Inset: horizontal axis extended to $100,000. The non-linearity is clearly visible at these scales.

This quantity is measured in "utils" per time, with the dimension "util" dependent on the utility function[4]. With the example parameter values and square-root utility this is 3.37 utils per month.

With insurance the (certain) rate of change of utility is

$$\langle r_u \rangle_{\text{own}}^{\text{in}} = \frac{U_{\text{own}}(W_{\text{own}} + G - F) - U_{\text{own}}(W_{\text{own}})}{\Delta t}, \tag{11}$$

or 3.46 utils per month with square-root utility.

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[4]The fact that the util is a dimension imposes restrictions on the utility function that are not generally recognised. Barenblatt (2003, p. 17 ff.) shows that any dimension function of any physically meaningful quantity, such as money, must be a power-law monomial, which restricts utility functions themselves to power-law monomials. This is a failure of the utility paradigm as it is often presented. The interpretation of $\ln(W)$ alone as a physically meaningful quantity is untenable – it should only ever occur as $\ln(W_1) - \ln(W_2) = \ln(W_1/W_2)$, where "utils" cancel out and the logarithm is taken of dimensionless quantities only. We note that von Neumann & Morgenstern (1944) pointed out that only differences between utilities may be considered meaningful, which avoids the problem.

The change in the rate of change of the expectation value of the owner's utility resulting from entering into the insurance contract is

$$\delta \langle r_u \rangle_{\text{own}} = \langle r_u \rangle_{\text{own}}^{\text{in}} - \langle r_u \rangle_{\text{own}}^{\text{un}},\qquad(12)$$

or 0.094 utils per month with square-root utility. In the example with square-root utility the owner should sign the contract, according to the utility paradigm.

Depending on the owner's utility function, U_{own}, and his wealth, W_{own}, buying insurance may increase or decrease the rate of change of the expectation value of his utility, with the result that the utility paradigm cannot say whether the contract should be signed or not. For almost 300 years this has been the celebrated classic resolution of the puzzle: since utility theory doesn't know what to do, it does not explicitly rule out the existence of insurance contracts.

The insurer's perspective

It has been said that the utility paradigm applies to individuals only, and not to companies. For example, Rothschild & Stiglitz (1976, p. 631) write: *"It is less straight-forward to describe how insurance companies decide which contracts they offer for sale [...]. We assume that companies are risk neutral and that they are concerned only with expected profits."* They continue in footnote 3: *"the theory of the firm behaviour under uncertainty is one of the more unsettled areas of economic theory, we cannot look to it for the sort of support [...] which the large body of literature devoted to the expected utility theorem provides."*

For reasons that will become clear in Section 5, especially 5.3, we don't follow Rothschild & Stiglitz (1976) and instead assign a utility function to the insurer, a procedure followed *e.g.* by Bernoulli (1738). To do this we assign to the insurer a large wealth, $W_{\text{ins}} = \$1,000,000$. We simply treat the insurer as an individual entity, and leave unspecified whether this is a natural person or a firm.

Without insurance then, the (certain) rate of change of the insurer's utility is

$$\langle r_u \rangle_{\text{ins}}^{\text{un}} = 0\qquad(13)$$

(the insurer does no business).

With insurance, the rate of change of the expectation value of the insurer's utility is

$$\langle r_u \rangle_{\text{ins}}^{\text{in}} = \frac{(1-p)U_{\text{ins}}(W_{\text{ins}} + F) + pU_{\text{ins}}(W_{\text{ins}} + F - L) - U_{\text{ins}}(W_{\text{ins}})}{\Delta t},\qquad(14)$$

or 0.043 utils per month with square-root utility.

The change in the rate of change of the expectation value of the insurer's utility resulting from entering into the contract is

$$\delta \langle r_u \rangle_{\text{ins}} = \langle r_u \rangle_{\text{ins}}^{\text{in}} - \langle r_u \rangle_{\text{ins}}^{\text{un}},\qquad(15)$$

0.043 utils per month with square-root utility. The different perspectives are summarised in Table 2. Without specifying the utility functions the utility

paradigm only makes weak predictions of behavioural consistency, based on assumed monotonicity (more money is better) and concavity (risk aversion) of the utility functions, as famously shown by von Neumann & Morgenstern (1944).

Table 2: Changes in the expectation value of utility for owner and insurer

$\langle r_u \rangle$	Owner	Insurer
insured	$\frac{U_{\mathrm{own}}(W_{\mathrm{own}}+G-F)-U_{\mathrm{own}}(W_{\mathrm{own}})}{\Delta t}$	$\frac{(1-p)U_{\mathrm{ins}}(W_{\mathrm{ins}}+F)+pU_{\mathrm{ins}}(W_{\mathrm{ins}}+F-L)-U_{\mathrm{ins}}(W_{\mathrm{ins}})}{\Delta t}$
uninsured	$\frac{(1-p)U_{\mathrm{own}}(W_{\mathrm{own}}+G)+pU_{\mathrm{own}}(W_{\mathrm{own}}-C)-U(W_{\mathrm{own}})}{\Delta t}$	0
difference	$\frac{U_{\mathrm{own}}(W_{\mathrm{own}}+G-F)-(1-p)U_{\mathrm{own}}(W_{\mathrm{own}}+G)-pU_{\mathrm{own}}(W_{\mathrm{own}}-C)}{\Delta t}$	$\frac{(1-p)U_{\mathrm{ins}}(W_{\mathrm{ins}}+F)+pU_{\mathrm{ins}}(W_{\mathrm{ins}}+F-L)-U_{\mathrm{ins}}(W_{\mathrm{ins}})}{\Delta t}$

5   Solution in the time paradigm

5.1   Introducing time – 20th vs. 17th-century mathematics

As we have argued previously (Peters, 2011b; Peters & Gell-Mann, 2014), changes in the expectation value of wealth are not *a priori* relevant to an individual. The expectation value is a mathematical object that has nice linearity properties which make it convenient for computations, but mathematical convenience is no reason for the object to be relevant. The physical operation that the expectation value encodes is this: take all possible events (the ship is lost or not), and create an ensemble of systems in numbers proportional to the probabilities of the events that occur in them. For example, create 100 systems, and let the ship travel safely in 95 of them, let it be lost in 5. Now take the different owners' wealths in all of these systems, pool them, and share them equally among the owners. In this setup, each owner receives with certainty $G - pL$, *i.e.* his gain minus the net premium.

Strictly speaking, this operation is an average over parallel universes (the different systems). Less strictly speaking, the operation corresponds to a large group of owners who make a contract with each other, whereby, at the end of the month, all wealths are pooled together, and shared equally.[5]

Thus, the operation of taking the expectation value is in essence the operation of signing an insurance contract – a perfect contract, no less, where all risk is eliminated and the premium is the net premium. No wonder, then, that in the expected-wealth paradigm there is no good reason for signing an insurance contract – mathematically, this has already happened!

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[5] See Peters & Adamou (2015) for a discussion of the same operation, carried our repeatedly, in the context of the evolution of cooperation.

Whichever way one chooses to conceptualise the expectation value, the central message should be clear: the rate of change of an individual's expected wealth bears no resemblance to, and is therefore a poor model of, how his wealth will evolve over time. The two mental pictures – many parallel cooperating trajectories *versus* a single trajectory unfolding over a long period – are not remotely concordant. We certainly cannot generally equate the performance of expectation values with the performance of a single system over time.

With this knowledge we can revisit the problem: clearly, an individual owner will not come to sensible conclusions if he compares his expectation value without buying insurance (which looks like he has signed a perfect insurance contract at the minimum fee that could possibly be offered) to his expectation value with buying insurance (which must be worse unless the insurer has made a mistake). As we will see, he *will* come to sensible conclusions if he uses the time-average growth rate of his wealth as a decision criterion.

5.2 Solution

Having argued that the treatment that led to the insurance puzzle is invalid because it assumes that humans optimise over imagined copies of themselves in parallel universes, it remains to show that the alternative treatment resolves the puzzle. To be specific: the alternative treatment is a different model of human decision making. The traditional model says "humans make decisions by optimising the expectation values of their wealths." Our model says "humans make decisions by optimising the time-average growth rates of their wealths." For concreteness we assume multiplicative dynamics[6]. Which model of human behaviour is better is only partly an empirical question[7]. Our model seems *a priori* more plausible to us because it does not introduce parallel universes, and because it resembles the real-world scenario it seeks to reflect. It is empirically strong because it resolves the insurance puzzle, and conceptually strong because all elements of the solution have clear physical significance.

Both the utility paradigm and the time paradigm consider the problem as it is stated to be underspecified. The utility paradigm assumes that people mentally convert money non-linearly into usefulness and then average linearly over all possible imagined outcomes; the time paradigm assumes that people mentally consider the future beyond the end of the single cargo shipment. Neither assumption is stated in the problem and each leads to a different type of solution, in which different choices must be made about the necessary additional information.

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[6]Multiplicative dynamics is a simple robust null model that solves the problem. We note that our treatment is not restricted to this special case but for clarity of exposition we refrain from a more comprehensive treatment.

[7]We explicitly disagree with Friedman (1953) on this point: the faithful reproduction of observed behaviour, even the successful prediction of not-yet observed behaviour, does not absolve a model from the requirement of using reasonable assumptions. Prediction and understanding are not the same thing. Rosenblueth & Wiener (1945, p. 316) point out that *"Not all scientific questions are directly amenable to experiment."*

This additional information corresponds to introducing an axiom. The time paradigm axiomatically assumes a stochastic dynamic of wealth. The utility paradigm axiomatically assumes a utility function. The utility function, which encodes human behaviour is further down the deductive chain – it can be derived from the dynamic. Therefore the time paradigm operates at a deeper level, and solutions in the time paradigm must be considered more fundamental. Another advantage of the time paradigm is that its axioms are more readily amenable to empirical verification: it is relatively easy to find out to what extent wealth is described by some dynamical model, whereas it is difficult to assess the inherently subjective value humans ascribe to changes in wealth.

The time paradigm postulates that humans optimise the time-average growth rate of their wealth. The usual procedure for computing such a time average is to first transform wealth in such a way as to generate an ergodic observable and then compute the expectation value of that observable. Being the expectation value of an ergodic observable, this will be the time average growth rate (Peters & Gell-Mann, 2014). Assuming multiplicative repetition, the time average growth rate is

$$
\begin{aligned}
\bar{g} &= \lim_{k \to \infty} \frac{1}{k \Delta t} \ln \left(\frac{W(t + k\Delta t)}{W(t)} \right) \\
&= \frac{\langle \Delta \ln W \rangle}{\Delta t} \\
&= \frac{\langle \ln W(t + \Delta t) \rangle - \ln W(t)}{\Delta t},
\end{aligned}
\tag{16}
$$

where the equality holds with probability 1. Equation (16) is simply the expectation value of the exponential growth rate, which is an ergodic observable that converges to a simple number under multiplicative dynamics, see (Gray, 2009, Chapter 7.7) for further discussion. Intriguingly, (Eq. 16) is identical to (Eq. 9) if we use logarithmic utility. Note that it is unnecessary to assume different utility functions for owner and insurer. The same logarithmic utility function for both makes the contract appear attractive to both according to the utility paradigm. This corresponds to both parties being subjected to the same dynamics. It turns out that utility functions correspond to specific dynamics, and the logarithm is twinned with multiplicative dynamics – the most ubiquitous dynamic in growing and evolving systems. The two paradigms are fundamentally different, however. For instance, in the utility paradigm if observed behaviour is found to be incompatible with logarithmic utility, the investigation ends. One simply concludes that the actors involved have a different utility function, but this does not point the way to a deeper understanding of the processes involved. On the other hand, in the time paradigm, if observed behaviour is found to be incompatible with multiplicative dynamics, this prompts the question whether the prevailing dynamic may be a different one and why that may be, or whether the actor may be making bad decisions.

Once more we go through the two parties' perspectives.

The owner's perspective

Without insurance, the time-average growth rate, under multiplicative dynamics, of the owner's wealth is

$$\bar{g}_{\text{own}}^{\text{un}} = \frac{1}{\Delta t} \left((1-p) \ln \left(\frac{W_{\text{own}} + G}{W_{\text{own}}} \right) + p \ln \left(\frac{W_{\text{own}} - C}{W_{\text{own}}} \right) \right), \qquad (17)$$

or 1.9% per month in our example.

With insurance, the time-average growth rate is

$$\bar{g}_{\text{own}}^{\text{in}} = \frac{1}{\Delta t} \ln \left(\frac{W_{\text{own}} + G - F}{W_{\text{own}}} \right), \qquad (18)$$

namely 2.2% per month in our example.

The change in the owner's time-average growth rate of wealth, resulting from entering into the insurance contract, is

$$\delta \bar{g}_{\text{own}} = \bar{g}_{\text{own}}^{\text{in}} - \bar{g}_{\text{own}}^{\text{un}}, \qquad (19)$$

here +0.24% per month.

Hence, the owner should sign the contract, according to the time paradigm. The rate of change of the expectation value of the owner's wealth decreases, (Eq. 4), but that's irrelevant because the owner is not an ensemble of owners sharing resources.

The insurer's perspective

The only difference between the insurer and the owner (apart from being at opposite sides of the contract) is in their wealths. Let's assume, as in Section 4, that the insurer's wealth is $W_{\text{ins}} = \$1,000,000$, namely ten times that of the ship owner.

Without insurance, the time-average growth rate of the insurer's wealth is

$$\bar{g}_{\text{ins}}^{\text{un}} = 0 \qquad (20)$$

because the insurer has no business.

With insurance, the time-average growth rate of the insurer's wealth is

$$\bar{g}_{\text{ins}}^{\text{in}} = \frac{1}{\Delta t} \left((1-p) \ln \left(\frac{W_{\text{ins}} + F}{W_{\text{ins}}} \right) + p \ln \left(\frac{W_{\text{ins}} + F - L}{W_{\text{ins}}} \right) \right), \qquad (21)$$

or 0.0071% per month in our example. The difference in time-average growth rates for the insurer is

$$\delta \bar{g}_{\text{ins}} = \bar{g}_{\text{ins}}^{\text{in}} - \bar{g}_{\text{ins}}^{\text{un}}, \qquad (22)$$

here +0.0071% per month. Hence, in the time paradigm not only the owner but also the insurer should sign the contract.

We consider this the fundamental solution to the insurance puzzle:

Both the owner and the insurer should sign the insurance contract, simply because this increases the time-average growth rates of both of their wealths.

This is consistent both with the observed existence of insurance contracts, and with human instincts about risk mitigation.

Because we've assumed the mental model of the dynamics to be purely multiplicative, only the relative risks matter: L, F, G and C only appear as proportions of current wealth W_{own} or W_{ins}. In this sense the value of a dollar is indeed inversely proportional to the wealth of the decision maker, as Bernoulli (1738) suggested on intuitive grounds. However, in the time paradigm the value of a dollar is given by the dynamic (the mode of repetition of the venture), not by psychology.

The insurance contract is seen to be beneficial to the owner and to the insurer – a mathematical impossibility under the expected-wealth paradigm. By taking time averages we find that the contract is a win-win situation, not a zero-sum game. This is possible because both parties are part of a non-stationary growth process. The utility paradigm often treats firms and individuals separately, see our quotes of (Rothschild & Stiglitz, 1976) in Section 4, but this is not necessary. We summarise the two perspectives in Table 3, and illustrate the effect in Fig. 3.

Table 3: Time-average growth rates for owner and insurer

$\bar{g}$	Owner	Insurer
insured	$\frac{1}{\Delta t} \ln\left(\frac{W_{\text{own}}+G-F}{W_{\text{own}}} \right)$	$\frac{1}{\Delta t}\left[(1-p)\ln\left(\frac{W_{\text{ins}}+F}{W_{\text{ins}}} \right) + p\ln\left(\frac{W_{\text{ins}}+F-L}{W_{\text{ins}}} \right) \right]$
uninsured	$\frac{1}{\Delta t}\left[(1-p)\ln\left(\frac{W_{\text{own}}+G}{W_{\text{own}}} \right) + p\ln\left(\frac{W_{\text{own}}-C}{W_{\text{own}}} \right) \right]$	0
difference	$\frac{1}{\Delta t}\left[\ln\left(\frac{W_{\text{own}}+G-F}{(W_{\text{own}}+G)^{(1-p)}(W_{\text{own}}-C)^p} \right) \right]$	$\frac{1}{\Delta t} \ln\left(\frac{(W_{\text{ins}}+F)^{(1-p)}(W_{\text{ins}}+F-L)^p}{W_{\text{ins}}} \right)$

Unlike in Table 1, the bottom row of Table 3 is not anti-symmetric – the change in time-average growth rate that the ship owner experiences when the contract is signed is *not* the negative of the insurer's change. A price range exists where both parties gain from entering into the contract. Here systemic risk is reduced and systemic growth supported: both parties will do better in the long run, which constitutes an explanation of the existence of an insurance market.
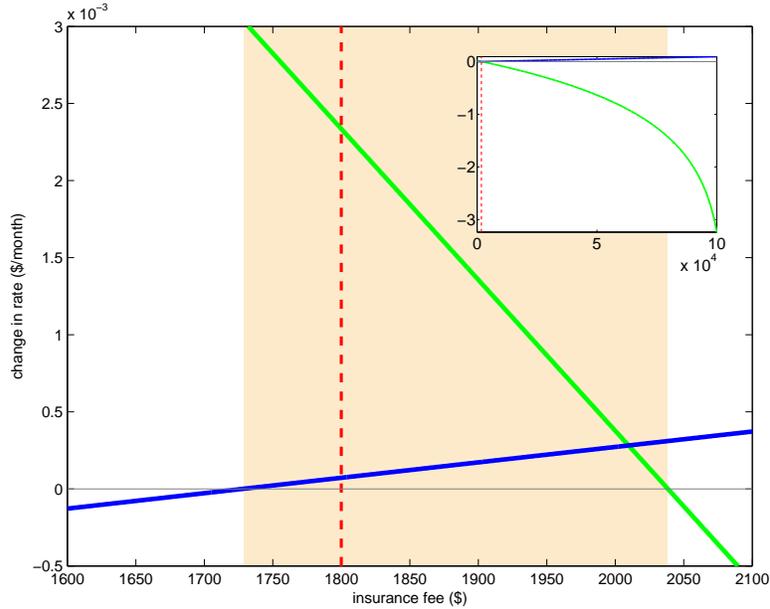
Figure 3: Change in time-average growth rate resulting from signing an insurance contract versus insurance fee. <u>Green line:</u> insurance buyer. <u>Blue line:</u> insurer. The vertical dashed line indicates the proposed insurance fee. <u>Shaded region:</u> Averaging over appropriate ergodic observables and without appealing to utility functions, a price range is seen to exist that is beneficial to both parties. Both the green and blue lines lie above zero. What is typically defined as the "fair price," the net premium, is detrimental to any insurer of finite wealth (the blue line is below zero at $1,700$ per month), leading to bankruptcy with probability 1. <u>Inset:</u> horizontal axis extended to $100,000$. The non-linearity is clearly visible at these scales.

5.3 The infinite-wealth limit – convergence to the expectation value

The time solution to the insurance puzzle can also be presented by considering the typical situation of an actual insurance company, such as a car insurer. Such a company may insure millions of cars, and therefore have access to a real-world ensemble that has nothing to do with parallel universes, just different cars.

If the pool of insured cars is large enough, the rate at which the insurer loses money to claims will be well described by the net premium, $(F - pL)/\Delta t$. This thought connects the time paradigm to the expected-wealth paradigm. Imagine an insurer whose wealth is much larger than any individual insured loss and any individual premium. In this case $\delta \bar{g}_{\text{ins}}$ of (Eq. 22) due to a single contract can

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be approximated by the first term of a Taylor expansion

$$\delta \bar{g}_{\text{ins}} = \frac{1}{\Delta t} \left[(1-p) \ln \left(1 + \frac{F}{W_{\text{ins}}} \right) + p \ln \left(1 + \frac{F-L}{W_{\text{ins}}} \right) \right]$$
$$= \frac{1}{\Delta t} \left[\frac{F-pL}{W_{\text{ins}}} + o \left(\frac{F}{W_{\text{ins}}}, \frac{F-L}{W_{\text{ins}}} \right) \right], \tag{23}$$

where we have used little-o notation. Multiplying the first-order term of (Eq. 23) with W_{ins} (an irrelevant constant at this order) yields exactly $\delta \langle r \rangle_{\text{ins}}$ of (Eq. 7). In other words, as the risk that is being insured becomes negligible to the insurer, the predictions of the time paradigm for the insurer's behaviour become increasingly similar to those of the expectation-value paradigm. The two mathematical models of human behaviour are identical in this limit, just as the models of Newtonian and Einsteinian mechanics are identical in the limit of small velocities, or the models of classical and quantum mechanics are identical in the limit as Planck's constant approaches zero (the "Correspondence Principle"). Similar behaviour was noted in the context of leveraged investments in continuous time: in the small-leverage limit, the rate of fractional changes of the expectation value converges to the time-average exponential growth rate (Peters, 2011*a*).

This explains why the expectation-value criterion is sometimes a reasonable description of large insurers' behaviour. Indeed, the assumption of risk neutrality for the insurance firm in (Rothschild & Stiglitz, 1976) is consistent with the time paradigm under multiplicative dynamics in the limit where the insurance fee and the insured loss are small compared to the wealth of the insurer. Outside this regime insurers that *"are concerned only with expected profits"* (Rothschild & Stiglitz, 1976, p. 631) will sooner or later go bankrupt because of excessive risk-taking. Any contract with $F > pL$ and $L > W_{\text{ins}} + F$ will appear attractive to a Rothschild-Stiglitz insurer ($\delta \langle r \rangle_{\text{ins}} > 0$) but the loss will bankrupt the firm when it occurs ($\delta \bar{g}_{\text{ins}} \to -\infty$).

Consequently, the behaviour outside the regime where the total underwritten risk is negligible compared to the insurer's resources is not well described by the maximisation of the expectation value of profits. Instead an actual insurance firm (or other risk-bearer) will set a limit for the acceptable probability of its bankruptcy over a given time interval, *i.e.* a risk tolerance, and explore the possible behavioural choices within the resulting constraints, see *e.g.* (Kaas *et al.*, 2008).

The difference apparent to Rothschild & Stiglitz (1976) between the owner (appearing to act according to a non-linear, concave, utility function) and the insurer (appearing risk neutral, *i.e.* appearing to act according to a linear utility function) is simply a consequence of scale. At large wealth the possible logarithmic changes in wealth are small enough for a linear approximation to the logarithm to be valid. The insurer of infinite wealth can accept the net premium (the ensemble average of the insured loss per time unit). The ensemble average is thus a limit that is often of practical relevance.

6 Conclusions: Business is when both parties gain

Figure 3 illustrates a remarkable difference to traditional actuarial thinking in economics. According to the time paradigm there is a win-win regime where both parties gain (both the green line and the blue line are above zero). The net premium, often called the "fair price" plays a less significant role – a range of prices is beneficial for both parties. The fair price is never part of that range (it is the end of the range in the limit of the infinitely wealthy insurer). Where exactly the price of trade will lie can be negotiated.

This seems to be a more general result: if it is true that people choose based on the physically sensible criterion of optimising growth over time, then a win-win range of prices will exist for any service or product that is traded. Business deals happen because both parties gain. This is the opposite of equilibrium thinking in economics, which tells us that deals happen either because one party cons or coerces the other into an agreement, or because both parties found the exact price where neither has a reason not to get involved.

Acknowledgments

Glossary

C Replacement cost of the ship. 4, 5, 7, 8, 11, 14, 15

$\delta \bar{g}_{\mathbf{ins}}$ Change in the time average (and expectation value) of the dynamic-specific growth rate of wealth (here exponential), experienced by the insurer when the insurance contract is signed. 14, 16, 17

$\delta \bar{g}_{\mathbf{own}}$ Change in the time average (and expectation value) of the dynamic-specific growth rate of wealth (here exponential), experienced by the owner when the insurance contract is signed. 14

$\delta \langle r \rangle_{\mathbf{ins}}$ Change in the rate of change of the expectation value of wealth, experienced by the insurer when the insurance contract is signed. 6, 7, 17

$\delta \langle r \rangle_{\mathbf{own}}$ Change in the rate of change of the expectation value of wealth, experienced by the owner when the insurance contract is signed. 5–7

$\delta \langle r_u \rangle_{\mathbf{ins}}$ Change in the rate of change of the expectation value of utility, experienced by the insurer when the insurance contract is signed. 10

$\langle r_u \rangle_{\mathbf{ins}}^{\mathbf{in}}$ Rate of change of the expectation value of the insurer's utility with insurance contract. 10

$\langle r_u \rangle_{\mathbf{ins}}^{\mathbf{un}}$ Rate of change of the expectation value of the insurer's utility without insurance contract. 10

$\langle r_u \rangle_{\mathbf{own}}^{\mathbf{in}}$ Rate of change of the expectation value of the insured owner's utility. 9, 10

$\langle r_u \rangle_{\mathbf{own}}^{\mathbf{un}}$ Rate of change of the expectation value of the uninsured owner's utility. 8, 10

t Time when the goods leave Amsterdam. 5, 8, 13

$U_{\mathbf{ins}}$ Insurer's utility function. 10, 11

$U_{\mathbf{own}}$ Ship owner's utility function. 8–11

U Utility function. 3, 8, 11

W Wealth. 3, 5, 8, 9, 13

$W_{\mathbf{ins}}$ Wealth of the insurer. 10, 11, 14, 15, 17

$W_{\mathbf{own}}$ Wealth of the insurance buyer. 5, 8–11, 14, 15

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