Pathways to Randomness in the Economy: Emergent Nonlinearity and Chaos in Economics and Finance

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by

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ABSTRACT

This paper: (1) Gives a general argument why research on nonlinear science in general and chaos in particular is important in economics and finance. (2) Puts forth two definitions of stochastic nonlinearity (IID-Linearity and MDS-Linearity) for nonlinear time series analysis and argues for their usefulness as organizing concepts not only for discussion of nonlinearity testing in time series econometrics but also for building a new class of structural asset pricing models. (3) Shows how to use ideas from interacting particle systems theory to build structural asset pricing models that turn IID-Linear or MDS-Linear earnings processes into non MDS-Linear equilibrium returns processes.

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1. INTRODUCTION

In the past few years a large literature on chaos and nonlinear science has appeared in economics. While the economics literature is large it is dwarfed by the parallel literature on chaos and nonlinear science in the other sciences. Here we will use the term "nonlinear science" to refer to the broader study of nonlinear dynamical systems, not just chaotic ones. More will be said about the domain of "nonlinear science" when we discuss journals and other outlets in the area.

A loose definition is this: "Nonlinear science" studies stochastic and deterministic dynamical systems that lead to "complex" dynamics. A deterministic dynamical system generates "complex dynamics" when "most" trajectories of the dynamical system do not converge to rest points or limit cycles. Here, in the stochastic case, "the dynamical system" refers to the underlying deterministic dynamical system, i.e. the system one obtains when the underlying stochastic forcing is shut off.

The main tasks of the current paper are three: (1) Give a general argument why research on nonlinear science in general and chaos in particular is important in economics and finance. (2) Put forth two definitions of stochastic nonlinearity (IID-Linearity and MDS-Linearity) for nonlinear time series analysis and argue for their usefulness as organizing concepts not only for discussion of nonlinearity testing in time series econometrics but also for building a new class of structural asset pricing models. (3) Show how to use ideas from interacting particle systems theory to build structural asset pricing models that turn IID-Linear or MDS-Linear earnings processes into non MDS-Linear equilibrium returns processes.

Although we give a sneak preview here the reader may wish to glance ahead at Section three for the concepts of IID-Linearity and MDS-Linearity. We call stochastic process $\{Y_t\}$ IID-Linear (MDS-Linear) if

$$Y_t - \mu = \sum \beta_j N_{t-j}, \ \sum \beta_j^2 < \infty,$$

where the innovations, also called shocks, N_t are Independent and Identically Distributed, abreviated IID, (form a Martingale Difference Sequence, abbreviated MDS). As we shall see in Section three, MDS-Linearity corresponds to the case where the conditional expectation of Y_{t+1} given $\{Y_t, Y_{t-1}, \ldots\}$ is a linear function of $\{Y_t, Y_{t-1}, \ldots\}$. The concept of IID-Linear is more stringent than MDS-Linear. Noisy chaos is a striking example of a process that is not MDS-Linear.

This paper is organized as follows. Section one contains a brief introduction. Section two uses this paper as a "bully pulpit" to make a plea for more research on nonlinear science in general and chaos in particular in

economics and finance. During this plea we give a very brief sketch of the literature. There is a sizable literature in economics on statistical testing for the presence of chaos and other nonlinearity in time series data. Since that has been covered elsewhere by articles in Benhabib (1992), as well as by Brock and Dechert (1991), Brock, Hsieh, LeBaron (1991), Sayers's article in Krasner (1990), and Scheinkman (1990), we shall say little about it here, except to say that many applications of the techniques found strong evidence against linear models driven by IID shocks and weaker evidence against a subclass of MDS-Linear models driven by certain parameterized forms of heteroscedastic shocks.

The techniques of Section three are purely statistical techniques for testing whether a time series sample comes from a linear process or whether a time series comes from a chaotic process. While statistical techniques are useful they are no substitute for a structural model in giving insights into the economic forces that may generate nonlinearity or chaos.

Section four develops structural models which can generate "endogenous" discontinuous changes in equilibrium asset prices. In particular we study a class of asset pricing models that generate returns per share processes that are not linear processes. The intent of these examples is to show how the theory of Section three can be used to build a parsimoniously parameterized econometrically and analytically tractable class of asset pricing models which allow returns data to speak to the presence of economic forces causing abrupt changes in volatility and returns. The models are also structured to have the potentiality of generating equilibrium returns that display GARCH effects (cf. Bollerslev etal., (1992)) as well as "excess volatility" relative to measured fundamentals.

2. THEORETICAL OVERVIEW

We shall deal with the theoretical part of the literature first. The journals (i) INTERNATIONAL JOURNAL OF BIFURCATION AND CHAOS (IJBC), (ii) JOURNAL OF NONLINEAR SCIENCE, (iii) PHYSICA D, (iv) CHAOS, (v) NONLINEARITY give a glimpse of impact that research on chaos in particular and nonlinear science in general has had in sciences other than economics. Indeed the term "nonlinear science" could be well defined to be the subject matter treated in the above journals. A good place for the reader to view this type of work in economics is the volume edited by Benhabib (1992).

An informal definition of chaos is this. A deterministic dynamical system is chaotic if it displays *sensitive dependence upon initial conditions* in the sense that small differences in initial conditions are magnified by iteration of the dynamical system. A stochastic dynamical system is *noisy chaos* if it is chaotic when the conditional variance of the stochastic driver (the ultimate source of the uncertainty) is set identically equal to zero.

The Benhabib book gives a guide to the literature on formal definitions of chaos as well as a multitude of theoretic economic models that show chaotic equilibria are theoretically possible and are non pathological. Indeed, in economies with many sectors sufficient conditions needed to obtain chaotic equilibria are not very strong when evaluated by the standards of general equilibrium theory.

Grandmont argues in a recent paper (1992): (i) the economic time series that display the most volatility, e.g., investment, inventories, durable goods, financial and stock markets, are those for which it appears that expectations play an important role in generating them, (ii) markets where expectations play an important role are most likely to be experience learning-induced local dynamic instability, (iii) plausible capital market imperfections, adjustment lags and limited substitutability can generate compex endogenous expectations-driven business cycles. He argues that it is

important to incorporate nonlinearities to study such fluctuations.

The recent book by Hommes (1991) shows how easy it is to produce chaos in Hicksian type models with lags in investment and consumption. Majumdar and Mitra (1992) locate sufficient conditions for robust ergodic chaos to appear in growth models. The studies cited above raise the key issue of the plausibility of chaos as a generating mechanism of fluctuations in the real economy.

Before we go further, I wish to discuss some issues, especially three common misunderstandings, that have been repeatedly raised to me while lecturing on the area of nonlinear science in general and chaos in particular.

I don't believe there is any disagreement amongst economists on whether exogenous shocks play an important role in astute modeling of economic fluctuations. The issue of contention concerns the relative value of modeling endogenous fluctuations directly to modelling a system driven by exogenous fluctuations, i.e., exogenous shocks. The issue whether chaos is an important source of endogenous fluctations is especially contentious for the case of aggregative macroeconomics (cf. Boldrin and Woodford's discussion of Sims's comment on Grandmont in Benhabib (1992)).

"Calibrationists" have criticised some theoretic models which produce chaotic equilibria for requiring parameter values that conflict with known measurements. Empirical work on testing for the presence of statistically detectable chaos in financial and macroeconomic time series data has not been very supportive of the hypothesis (cf. Ramsey, Sayers, and Rothman (1988)). This controversy has lead to some misunderstandings on the importance of research on chaotic and other nonlinear phenomena in economics.

The first misunderstanding is this. Just because evidence for chaos in time series data is weak does not mean that chaos is not a useful lens through which to view economic activity. The joint problem of data quality and weakness of statistical tests make the power of such tests to detect chaos in economic data particularly weak.

Indeed a recent paper by Barnett, Gallant, Hinich, and Jensen (1992) applied three tests for nonlinearity and chaos to monetary data and found inconsistent results across the three tests. They state: "Given the weak nature of that hypothesis and the implausible nature of the alternative--that the explanation of fluctuations lies in supernatural shocks to a linear universe--we find the degree of controversy regarding the existence of nonlinearity or chaos in economic data to be surprising." This statement seems to me to be right on target. Even if the reader does not agree with Barnett etal. it seems more productive to adopt a scientific research program that directs one to search for a mechanism that generates the observed movements in time series data which minimizes the role of "exogenous shocks."

The second misunderstanding is to conclude that weak evidence for chaos implies weak evidence for nonlinearity in general. Chaos is a very special species of nonlinearity. Methods inspired by the attempt to detect chaos have turned out to be useful in detecting other types of nonlinearity. There is another reason to be nervous about the use of linear methods in macroeconomics.

The reader should be reminded that the currently available sufficient conditions on stochastic multisector models for convergence to a unique stochastic steady state are severe (cf. Marimon (1989)). Much of modern macro economics, including real business cycle theory, is built upon the foundations of models that have a unique globally asymptotically stable stochastic steady state. The cases where linear approximation methods (after appropriate transformations) work well are, for the most part, the cases where attractors are simple points or cycles (when the driving noise is shut off). So theory is no refuge for the linearist.

The third, and probably the most important misunderstanding is to

conclude that nonlinearity is unimportant in macroeconomics and finance because out-of-sample prediction of nonlinear models does not appear to be better than linear models such as random walk models in finance. Prominant examples of studies that find no out-of-sample forecast improvement for nonlinear models are Diebold and Nason (1990), Meese and Rose (1991). Perhaps, because of these negative results on forecasting, some are lead to question the value of research on nonlinear econometric models in the times series area.

However, LeBaron (1992a,b) has found reliable out-of-sample nonlinear forecast improvements in stock returns by cleverly conditioning on local information such as local volatility. He shows that measures of near future predictability increase when measures of near past local volatility fall. Antoniewicz (1992) obtains forecast improvements on returns for individual stocks by conditioning on local volume by use of certain trading strategies.

The main point is this. Earlier studies examined unconditional measures of out-of-sample forecast improvment. Estimates of these measures are an average over the sample over periods where the forecast may be doing well and where it may be doing poorly. LeBaron shows that this averaging can make it difficult to discover conditioning information which could help identify periods when out-of-sample forecast improvement is possible.

Since Brock (1991b) gave an heuristic argument that tests in the family studied by Brock, Dechert, Scheinkman, LeBaron (1990) and de Lima (1992a,b) are good at exploring the whole space for local pockets of predictability therefore a rejection of IID-Linearity by one of these tests suggests that effort should be made to detect potential pockets of predictability. LeBaron's work can be viewed as a successful location of such pockets of predictability. The trading rule specification tests of Brock, Lakonishok, and LeBaron (1992) are also designed to locate zones where prediction might be possible.

Since LeBaron is working in the area of finance where the Efficient Markets Hypothesis gives a strong argument that any predictability is going to have to be subtle to prevent traders from exploiting it, therefore success at finding prediction possibilities in this area suggests that search in other areas of economics might be even more fruitful. Having dealt with some concerns about techniques inspired by nonlinear science in economics and finance let me turn to an overview of interest in nonlinear science in disciplines other than economics and finance.

My reading of natural science literature suggests, after initial debate on the claims of having found actual evidence for chaos in Nature, that natural science accepts the usefulness of nonlinear science in general. Evidence for this view follows.

First, a U.S. National Academy of Sciences report states, "As a consequence of its fundamental intellectual appeal and potential technological applications, nonlinear science is currently experiencing a phase of very rapid growth....In any effort to guide this research, however, it is imperative that nonlinear science be recognized for what it is: an inherently interdisciplinary effort..." (NAS (1987, p. 14)). The report worries about the difficulty of supporting research in this area within the confines of the balkanized U.S. university department system whose reward structure tends to discourage bold interdisciplinary research. They also worry about the large amounts of support of the area in other countries relative to the support in the U.S. They conclude that nonlinear science has "...a remarkable breadth of application and the potential to influence both our basic understanding of the world and our daily life."

A second piece of evidence is a dramatic bar graph in Casti (1992, Vol. I, p.viii) where he plots the number of articles on chaos and fractals by year from 1974-1990. The bar graph shows an explosion of interest starting in 1983

which is rapidly growing to a level of almost four thousand articles in 1990. Since nonlinear science covers the general species of complex nonlinearity and since chaos and fractals are subsets of the area, Casti's bar graph understates the true extent of activity in this area.

Here is my attempt at distillation of a general view which has emerged from a huge literature in natural science. Natural science work on chaos leads to the view that dynamical systems which are composed of many locally and/or globally interacting parts with a variety of lag lengths due to adjustment dynamics or other sources of delayed reaction are quite likely to be chaotic where "likely" is measured relative to a population of general dynamical systems.

In practice measurements taken on the output of such systems are usually aggregative and corrupted by noise. Therefore even though the underlying generating mechanism may be chaotic the measurements taken on the system appear to be stochastic or purely random. In order to see how tough it can be for statistical tests to detect patterns in some deterministic dynamical systems take a look at Griffeath's comment (especially his reference to Wolfram's work on cellular automata) in Berliner (1992).

A prototypical example in natural science is fluid flow dynamics (cf. Van Atta's article in Krasner (1990)). For an economist fluid flow dynamics may, perhaps, be usefully viewed as a cellular automaton defined on a large dimensional state space. In certain Taylor-Couette fluid flow experiments (where the fluid is "weakly" turbulent) velocity measurements of a small chunk of fluid appear stochastic to many statistical tests but statistical tests based upon chaos theory detect evidence of low dimensional chaos.

Studies in epidemiology are discussed by Schaffer in Krasner (1990). Here, much as in economics, the controversy centers around whether, for example, the time series of measles cases is better described as a low order autoregression with seasonalities associated with the opening and closing of schools or is better described by a periodically forced dynamical system with a delay structure across components, perhaps along the lines of Kuznetsov etal. (1992), which can take the torus destruction route to chaos. The working conditions in epidemiology and biology are closer to those in economics where data quality is not so high and where laboratory experiments are expensive or impossible.

It appears that chaos is useful as a lens through which to view the world in epidemiology, biology, and ecology, not because it helps so much in prediction but because it is <u>suggestive of pathways to complex dynamics</u>.

This type of viewpoint leads to a paradigmatic shift in thinking about useful methods of study of such fields. Some scientists have been taking the view that in many cases linearization methods are suspect and the only excuse for using them is computational cost. Advances in computation have removed this constraint. Indeed some natural scientists are becoming rather sceptical about linearization. See, for example, Chua's editorial in INTERNATIONAL JOURNAL OF BIFURCATIONS AND CHAOS, March, 1991.

In view of the rapid increase in nonlinear science activity in the other sciences, and, with the dramatic decline of computer costs making nonlinear science research within the realm of any researcher with a PC, one might argue that economics ignores nonlinear science at its peril.

Indeed people of a more practical sort with no incentive to have allegiance to any particular academic methodology have been recently using ideas from nonlinear science such as genetic algorithms and neural nets to design trading strategies for financial assets. Three examples that have recently hit the popular media are Hawley etal. (1990), LONDON ECONOMIST, August 15, 1992, p. 70, and "The New Rocket Science Hits Corporate Finance," BUSINESS WEEK, Nov. 2, 1992. Reading between the lines one can see that at least one of the strategies discussed by the ECONOMIST and BUSINESS WEEK was

inspired by Holland's (1992) "bottom up" approach to artifical intelligence by creation of an artificial ecology of strategies encoded by bit strings so that evolutionary Darwinistic dynamics can be simulated via computer.

In this system the best strategies are those which survive many generations of simulated evolutionary struggle. The Santa Fe Institute has stimulated research along this line in economics. Prominant examples are Anderson, Arrow, and Pines (1988), Arthur (1992), and Sargent (1992). Arthur (1992) and Sargent (1992) contain elegant statements of this approach to modeling "bounded rationality" in economics.

More on the Santa Fe theme can be found in a recent SCIENTIFIC AMERICAN article, "The Edge of Chaos: Complexity is a Metaphor at the Santa Fe Institute," October, 1992. The Santa Fe Institute studies complex dynamical systems and uses them as an organizing theme to study a catalogue of phenomena including the economy. See Anderson, Arrow, and Pines (1988) for an early statement of the Santa Fe approach. While I believe that there is a general consensus in economics that research in economics in the general area of nonlinear science as exemplified by the Santa Fe Institute is valuable, the usefulness of research on the particular area of chaos may not have such a consensus.

Nevertheless I argued above that this kind of research has been important. Other reasons why the research is important are these. First, in models with many sectors with a variety of adjustment lags it is easy to produce chaotic equilibria for plausible parameter values. Yet it is easy to produce examples where the aggregates do not appear chaotic to statistical tests for chaos. So aggregation may be responsible for the lack of evidence of chaos in macroeconomic data.

Second, the article by McNevin and Neftci in Benhabib (1992) argued that a set of aggregate data is less anti-symmetric than the disaggregated data under plausible economic conditions. Anti-symmetry is evidence consistent with nonlinearity because symmetric input into a linear map leads to symmetric output. They argue that the cyclical behavior of major capital goods industries is likely to be out of phase at business cycle frequencies and this would lead to symmetric aggregates even though the components are anti-symmetric. Their evidence is consistent with this story.

This situation is rather similar to the work of Sugihara and May (1990). They exhibit evidence consistent with the view that aggregate data on measles looks like an AR(2) with seasonalities associated with the opening and closing of schools is composed of components which behave in a manner more consistent with chaos (Sugihara, Grenfell, and May (1990)). Indeed when Sugihara, Grenfell, and May (1990) disaggregated the data they found evidence that there was a lag structure in propagation of the disease from area to area which generated dynamical information consistent with chaos. Note that we are *not* saying they showed the data was chaotic. We are only saying that the disaggregated data exhibited behavior consistent with chaos.

Third, research on chaos has sensitized scholars to pathways for emergent structure such as emergent nonlinearity. It is important to recall that chaos is a very special form of nonlinearity and, hence, the set of nonlinear data generating processes is much larger than the set of chaotic data generating processes.

3. TESTING FOR CHAOS AND GENERAL NONLINEARITY

A common method, but certaintly not the only one, of testing for "neglected structure" of any form is to estimate a best fitting model in a given null hypothesis class and pass the estimated residuals through a testing procedure designed to detect "neglected structure." If the null hypothesis class is the linear class this gives a procedure to test for nonlinearity.

In order to discuss this subject we need some definitions which we take from Brock and Potter (1992). For brevity we concentrate on scalar valued processes and q lags in the law of motion (1.b) below.

Definition 1: We say the observed data process $\{A(t)\}$ is generated by a noisy deterministically chaotic explanation, "noisy chaotic" for short, if

(3.1.a) $A_t = h(X_t, M_t)$,

(3.1.b) $X_t = G(X_{t-1}, \ldots, X_{t-q}, V_t)$,

where $\{X_t\}$ (when $V_t = 0$) is generated by the deterministic dynamics,

(3.1.c) $x_t = G(x_{t-1}, \ldots, x_{t-q}, 0)$,

which is *chaotic*, that is to say the largest Lyapunov exponent (defined below) exists, is constant almost surely with respect to the assumed unique natural invariant measure of $G(\cdot)$, and is positive. □

Here $\{M_t\}$, $\{V_t\}$ are mutually independent mean zero, finite variance, Independent and Identically Distributed (IID) processes. Here $\{M_t\}$ represents measurement error, $h(x,m)$ is a noisy observer function of the state X_t, and $\{V_t\}$ is dynamical noise.

We warn the reader that positive largest Lyapunov exponent of the underlying deterministic map is not the only definition of chaos which appears in the literature. However this definition and, all definitions we have seen, share the following hallmark of chaos: Sensitive Dependence upon Initial Conditions (SDIC). Turn now to a definition of largest lyapunov exponent.

Definition: Largest Lyapunov Exponent of map F(x).

Let $F:\mathbb{R}^n \to \mathbb{R}^n$. The largest Lyapunov exponent, λ is defined by

(3.2) $\lambda \equiv \lim_{t \to \infty} \ln [\, D_{x_o} F^t . v \,]/t$,

where D_{x_o}, ".", v, ln, F^t, . denote derivative w.r.t initial condition x_o at time zero, matrix product with direction vector v, natural logarithm, map F applied t times (the t-th iterate of F), and matrix norm respectively.□

The following well known scalar valued example, called the tent map,

(3.3) $F(x)=1-|2x-1|$,

is a deterministic chaos with the following properties: F(x) maps [0,1] to itself, and for almost all initial conditions, $x_o \in [0,1]$, w.r.t., Lebesque measure on [0,1], the trajectory $x_t(x_o)$ of the dynamics, (1.c) is second order white noise i.e., has flat spectrum, and, the AutoCorrelation Function (ACF) is zero at all leads and lags. The largest Lyapunov exponent is $\lambda=\ln(2)>0$. There are many examples of deterministic chaoses. They share the feature that they are not predictable in the long term but they are predictable in the short term.

The approach of Barnett, Gallant, Hinich, and Jensen (1992) locates sufficient conditions on the above setup such that the method of delays can be used to "reconstruct" the underlying deterministic dynamical system (3.1c) so

that nonparametric regression can be used to obtain a consistent estimator of (3.1c) so that a consistent estimator of the largest Lyapunov exponent can be found. Once a consistent estimator of the largest Lyapunov exponent is in hand, they test whether it is positive.

The definition of chaos as positive largest Lyapunov exponent naturally leads to an heuristic suggestion why chaotic dynamics should be expected for the dynamics $x_{t+1}=f(x_t)$, $f:\mathbb{R}^n \to \mathbb{R}^n$ for n large enough. The intuitive idea is this: If a system f is "drawn at random" the chances of getting one with a positive Lyapunov exponent should tend to one as $n \to \infty$.

In order to see why it may be possible to formulate and prove such a result turn to Ruelle's (1989, Chapter 9) treatment of Liapunov exponents. Place enough restrictions on each dynamical system f(.) so that the Oseledec Multiplicative ergodic theorem can be applied to give existence of the limit of the 1/2N-th root of the product of the derivative of the N-iterate of f(.) with its adjoint. Call this limit Λ_x for initial condition x. The logarithms of the eigenvalues of this limit matrix are the Liapunov exponents. While the limit exists for ρ-almost all initial conditions, the invariant measure ρ which appears in the theorem depends upon f(.). Also the measure ρ may contain "atoms", i.e., may not be absolutely continuous with respect to Lebesque measure. Hence there are obstacles on the route to showing that the story we tell below could serve as a metaphor for the likelihood of drawing a dynamical system whose limit matrix Λ_x exists and which has an eigenvalue with modulus greater than one, i.e., a positive Liapunov exponent, i.e, the dynamical system is chaotic.

Consider the following story which will serve as a kind of metaphor. Let B be a large positive number. Draw n numbers λ_i, $i=1,2,\ldots,n$ at random from the set $[-B,B]$ according to cumulative probability density distribution function $P_i(.)$ for the ith draw. Let these numbers play the role of the eigenvalues of Λ_x above. The probability that at least one λ_i is greater than one (i.e. we have a positive largest Lyapunov exponent for the linear dynamics on $\mathbb{R}^n$ given by $x_{j,t+1}=\lambda_j x_{jt}$, $j=1,2,\ldots,n$) is one minus the probability that all λ_i are less than or equal to one. Assuming independent draws we see immediately that $A_n \equiv \Pi_{i=1}^{i=n} \Pr\{\lambda_i \le 1\}$ is nonincreasing in n, hence converges to a nonnegative limit L, as $n \to \infty$. If $L>0$ then taking logs shows us that $\log[\Pr\{\lambda_i \le 1\}] \to 0$, $i \to \infty$, i.e., $\Pr\{\lambda_i \le 1\} \to 1$, $i \to \infty$. This gives us

<u>Proposition</u>: Let $\{P_i\}_{i=1}^{\infty}$ be a family of distribution functions such that $\liminf_{i \to \infty} \Pr\{\lambda_i \le 1\} < 1$. Then as $n \to \infty$ the probability that at least one $\lambda_i > 1$ in n draws converges to unity.■

We hasten to add that the above argument is only meant to suggest that it is not absurd to expect that chances are high for obtaining a positive Lyapunov exponent for a dynamical system on $\mathbb{R}^n$ "drawn at random" (however "drawn at random" is given precise meaning). We consider it an interesting research project to find sufficient conditions on the space of dynamical systems on $\mathbb{R}^n$ so that the likelihood of a chaotic system could be made precise as n tends to infinity. At this stage we are simply trying to show that it is not implausible to expect that a "lot" of systems are chaotic if n is large enough. Turn now to a treatment of general nonlinearity that goes beyond chaos in particular and nonlinearity of deterministic dynamical systems in general.

SOME NOTIONS OF STOCHASTIC LINEARITY AND NONLINEARITY

For brevity we consider scalar valued strictly stationary stochastic processes. Consider the following stochastic process

$$(3.4) \quad Y_t - \mu = \sum \beta_j N_{t-j}, \ \sum \beta_j^2 < \infty,$$

where $\{N_n\}$ is a mean zero, finite variance denoted, $(0,\sigma^2)$, strictly stationary stochastic process. In the discussion here $\sum$ ranges 0 to ∞. It can be generalized.

We discuss two commonly used definitions of stochastic linearity: MDS-Linear and IID-Linear.

Definition (MDS-Linear) (Hall and Heyde (1980, p. 182, 183)):

The stochastic process $\{Y_t\}$ is MDS-Linear if it can be represented in the form (3.4) above where the "innovations" $\{N_t\}$ are a Martingale Difference Sequence (MDS) relative to the sigma algebras $\mathbb{F}_t$ generated by $\{Y_s, s \le t\}$.□

Hence a stochastic process is "MDS-Linear" if it can be represented as a linear filter applied to MDS innovations. To put it another way, the best Mean Squared Error (MSE) predictor based upon the past is the same as the best linear predictor based upon the past.

De Jong (1992) shows how the Bierens consistent conditional moment test of functional form can can be adapted to create a consistent test of MDS Linearity. The intuitive idea is to consistently estimate (under the null hypothesis of MDS-Linearity) a linear model and pass the residuals through De Jong's adaptation of Bierens's test. We refer the reader to De Jong for the details. Turn now to the definition of IID-Linear.

Definition (IID-Linear) (Hall and Heyde (1980, p. 198)).

The stochastic process $\{Y_t\}$ is IID-Linear if it can be written in the form (4) where the innovations $\{N_t\}$ are IID $(0,\sigma^2)$.□

The test of Brock, Dechert, Scheinkman, LeBaron (1990) is especially adaptable to testing the hypothesis of IID-Linearity. This is so because Brock and Dechert (1991), Brock, Dechert, Scheinkman, and LeBaron (1990), Brock, Hsieh, and LeBaron (1991) show that the first order asymptotic distribution on the estimated residuals of best fitting linear models are the same as on the true residuals for a large class of IID-Linear processes. The last two references argue this point both by theory and monte carlo work. De Lima (1992a,b) gives the most general and most complete proof of this invariance property for a family of related statistical tests.

Note that the Wold representation theorem says any purely (linearly) nondeterministic stochastic process has a representation of the form (4) for some $\{N_t\}$ innovation process which is *uncorrelated* (HH (p. 182)). The two definitions of linearity require much more than mere uncorrelatedness of the innovations. That is what gives the definitions content. Futhermore requiring (as in the concept of MDS-Linear) that the best MSE-linear predictor be the best MSE-predictor seems to be as far as one can go in weakening the IID requirement on $\{N_t\}$ without running into the inherent nontestability of the Wold decomposition.

The above exposition gives an heuristic overview of the two main definitions of stochastic linearity. However, in financial applications it is controversial to assume that second order moments exist of outputs and innovations. The reason is simple. There is strong evidence that the unconditional variance of asset returns is infinite and, furthermore, conditional volatility measures are extremely persistent (cf. Loretan and Phillips (1992) and their references). For this reason the definitions require relaxation of the moment conditions.

De Lima (1992a,b) provides a general class of tests which can be used to test the hypothesis of IID-Linear under minimal moment restrictions. Essentially de Lima requires no more moment requirements than those needed to consistently estimate linear models. Furthermore the first order asymptotic distribution of his tests on estimated residuals are rigorously shown to be independent of the estimation procedure for a large class of IID-Linear data generating processes. Furthermore he shows by theory and monte carlo work that moment requirements of rival tests matter for correct inference under conditions typical for financial data. We urge the reader who works with heavy tailed data generating processes such as those in finance to read de Lima's two papers.

We hasten to add that the literature on testing for nonlinearity and estimation of nonlinear models is vast and that the point of view exposited here disproportionately represents my own work. The book by Brock, Hsieh, and LeBaron (1991) expounds the point of view taken here and briefly attempts to relate it to other parts of the econometric literature. The books by Casdagli and Eubank, (1992), Granger and Terasvirta (1992), and Tong (1990) should be consulted by the reader for a more balanced treatment of nonlinear time series econometrics.

Turn now to development of new classes of structural asset pricing models that generate non MDS-Linear equilibrium processes for returns per share of risky asset.

Recall that MDS-Linearity is equivalent to: The best MSE predictor given the past, i.e., the conditional expectation given the past, is the best linear MSE predictor. Hence any class of models that contain endogenous jumps and discontinuities in response to changes in the variables used for prediction cannot be MDS-Linear because linear predictors are continuous functions of the variables used for prediction.

4. STRUCTURAL MODELING USING INTERACTING PARTICLE SYSTEMS THEORY

In this section we exhibit a class of asset pricing models that show how MDS-Linear earnings processes can be transformed into equilibrium returns per share processes that are not MDS-Linear. While we emphasize that more conventional asset pricing theories such as Lucas (1978) and Brock (1982) can transform linear earnings processes into nonlinear returns processes through the market equilibration equations, in these models small changes in the environment do not lead to large changes in returns or returns volatility. Evidence in articles such as Haugen etal. (1991) suggests that abrupt changes in returns and volatility which are difficult to link to measures of fundamentals are quite common. We want our models to be able to address such evidence. Turn now to a class of models that endogenize discontinuous responses to changes in the environment and history of evolution of the system.

We shall use the probability structure of interacting particles systems (IPS) theory as an input into building our class of asset pricing models. See Durlauf (1989a,b, 1991a,b) and his references, especially to Föllmer, for uses of IPS theory in economics. Here we shall complement this work by fusing together ideas from discrete choice theory (e.g. Manski and McFadden (1981), and IPS by using mean field theoretic arguments to obtain closed form solutions for equilibria in our models in the large economy limit. In this way we can formulate the theory at a level of accuracy sufficent to capture the phase transition behavior emphasized by Durlauf, but still have the convenience of closed form solutions which can be adapted for statistical inference.

Hence, this part of our paper is a methodological in the same sense as Lucas (1978). The emphasis will be on finding parsimoniously parameterized, yet flexible, probability structures. The modelling technique offered here will be applied to examples in order to show its usefulness.

The organization of Section four of this paper is as follows. First, in Section 4.1, we state the general probability structure of interacting systems that we shall use. Second, in Section 4.2 we shall apply this probability structure to develop asset pricing models where demands are cross-dependent at a point in time over the set of traders. The large economy limit will be taken and conditions will be located on the strength of the cross dependence for the cross sectional ergodic theorem to hold. We shall then study the temporal evolution of the cross sectional dependence. The models are framed to be econometrically tractible to adaptions of the method of moments.

In Section 4.2 we treat the first example of our type of model. This is a formalization of "noise trader" models in economics and finance, where we find sufficient conditions on the probability structure for the noise traders to matter in the large economy limit. Since, "noise" trader models are controversial we emphasize at the outset that our type of model may be interpreted as a model where traders have heterogeneous beliefs or heterogeneous estimation or learning methods for relevant conditional moments needed to form their demands for assets. The new ingredient that we add is a parameterization of the cross-dependence of the heterogeneity that is econometrically tractable and leads to the uncovering of sufficient conditions for the heterogeneity to matter in the large economy limit.

In Section 4.3 we develop an asset pricing model where dependence of each trader's income on the market portfolio is itself dependent across the set of traders. This model leads to a simple relationship for the equilibrium price of the risky asset and relatively simple equilibrium volume dynamics.

Section 4.4 treats a version of Campbell, Grossman, and Wang's (1991), hereafter, CGW, model of traders with random risk aversion parameters. In our version the temporal movement of risk aversion evolves endogenously in such a

way that explosive bursts of volatility are possible in a rational expectations equilibrium. Our model is a nonlinear model that nests the CGW model as a special case. We indicate how the parameters of the model may be estimated using data on price and volume.

None of the above models are rational expectations models with asymmetrically informed agents in the sense of Gennote and Leland (1990), Hellwig (1980), (1982). In Section 4.5 we briefly show how Hellwig's (1980) large economy limit theorem can be used to produce a model with an equilibrium price relationship which can display abrupt changes to small changes in the environment.

Section 4.6 shows how to build a simple macro finance asset pricing model that "endogenizes" the exogenous shocks in the Lucas (1978), and Brock (1982) models. This example was stimulated by Durlauf (1991a,b).

These models illustrate that interactive systems probability modelling can produce analytically tractable asset pricing and macroeconomic models. The models all share the common property that small input noise into the environment can produce large noisy movements in equilibria. The models suggest economic pathways through which input noise magnification can occur.

GENERAL PROBABILITY STRUCTURE

We exposit the simplest version of our probability structure here. The appendix contains a generalization where the interactions are be considered over disjoint sets $A_1,\ldots,A_K$ where types are homogeneous within each set but heterogeneous across each set. In the Appendix, the large system limit (as N≡total number-->∞) is taken by holding the fraction of each type k=1,2,...,K constant.

To formalize the simplest version which contains the main ideas, let Ω be a set of real numbers, let Ω_N be its N-fold Cartesian product, $\omega\varepsilon\Omega_N$, and put

$$(4.1.1)\quad Pr\{\omega\}=\exp[\beta G]P_N(\omega)/Z,\ G\equiv U(\omega)+(1/2)\textstyle\sum_i\sum_{(i)}J_{ij}\ \omega_i\omega_j+h\sum\omega_i,\ U(\omega)\equiv\sum u(\omega_i)$$

where $\sum_i$ is over i=1,2,...,N, $\sum_{(i)}$ is over "neighbors of i", "$Pr\{\omega\}$" denotes probability of social state ω, $u(\omega_i)$ is own utility to agent i of choice $\omega_i\varepsilon\Omega$, $Z\equiv\sum\exp[\beta G(\nu)]P_N(\nu)$, $\sum$ is over all ν, and β is a parameter whose role will be explained later. Here $P_N(\nu)$ denotes the product probability on Ω_N induced by the common distribution function F on Ω. We will concentrate on the case where Ω is finite and F is a sum of "dirac deltas" but use $\sum$ and $\int$ interchangeably to suggest the natural extension to a continuous state space.

The best way to think about this structure is to think of (4.1.1) as giving the joint distribution of social states ω of a society of N individuals, each facing a choice from a set of alternatives, Ω. Here J_{ij} is a measure of the strength of interaction between individuals i,j located at sites i,j. We wish to exposit a discrete choice (Manski and McFadden (1981)) interpretation of (4.1.1) because this will be important to our development.

Consider the discrete choice model

$$(4.1.2)\quad V(\omega)=G(\omega)+\mu\varepsilon(\omega),\ \{\varepsilon(.)\}\ \text{IIDEV}.$$

Here "IIDEV" denotes Independent and Identically Distributed Extreme Value. The model (4.1.2) represents a stochastic social utility model where the errors $\varepsilon(.)$ are IID extreme value (Weibull) over ω in Ω_N. Manski and McFadden

(1981, pp. 198-272) show that the probability that a particular social state ω is social utility maximizing is given by (4.1.1) with $\beta \equiv 1/\mu$, where β is called the "intensity of choice." Note that $\beta=0$ gives the most random measure across social states, i.e., each social state has probability $1/|\Omega_N|=1/|\Omega|^N$, where $|\Omega|$ denotes the cardinality of the finite set Ω.

Anderson, de Palma, and Thisse (1993), Chapter 2)), hereafter "ADT," review results in the discrete choice literature that show $E\{\max G(\omega)\}=(1/\beta)\ln(Z)$ where "ln" denotes the natural logarithm and "max" is over $\omega \varepsilon \Omega_N$. This gives a nice connection between the welfare measure, $E\{\max G(\omega)\}$, of discrete choice theory and the free energy function of statistical mechanics. They are the same except for a change in sign. See Kac (1968) for the free energy function. ADT also show how $(1/\beta)$ is related to measures of diversity in differentiated product models as well as to the CES parameter in Constant Elasticity of Substitution differentiated product models. It is helpful to keep these possible interpretations of $(1/\beta)$ in mind while reading the sequel and to keep in mind that it does not have to be interpreted as "inverse temperature" which is the standard interpretation in statistical mechanics.

We wish to relate individual choices to aggregate social choice. We wish to compute long run averages and locate conditions for ergodicity failure for probability systems like (4.1.1) and (4.1.2). The interacting particles systems theory, hereafter "IPS", discussed by Durlauf (1989a,b; 1991a,b), Ellis (1985) and their references is the tool we use. Durlauf's work locates sufficient conditions for ergodicity failure for models with general $\{J_{ij}\}$. We specialize here to a rather coarse level of approximation, called "mean field theory," which replaces the joint probability distribution in (4.1.1) with an approximating product probability.

This level of approximation is accurate enough to (i) uncover sufficient conditions for phase transitions which predict phase transition behavior in the more general case, (ii) give useful parameterizations for economic modelling that yield econometrically tractable models, (iii) give the same equations for limit values of certain bulk quantities such as means as more general structures. We shall explain below.

Let $\langle\omega_i\rangle$ denote expected value computed w.r.t probability structure (4.1.1). Assume "translation" invariance, $\sum_{(i)} J(i-j)\omega_j = \sum_{(k)} J(k-j)\omega_j$, for all i,k. This implies $m \equiv \langle\omega_i\rangle = \langle\omega_k\rangle$ for all i,k. Consider the component of social utility in (4.1.1) that is generated by agent i,

$$(4.1.3)\quad V_i(\omega_i;\omega_{-i}) \equiv u(\omega_i) + [\textstyle\sum_{(i)} J(i-j)\omega_j]\omega_i + h\omega_i + \mu\varepsilon(\omega_i),\ \{\varepsilon(\omega_i)\}\ \text{IIDEV}.$$

We remark that in later applications, as in Durlauf (1991a,b), we shall interpret ω_{-i} as the previous period choices of agents other than i. Furthermore the utility function u(.), the parameters J(i-j), h, μ, and the distribution of ε(.) can depend upon the past. For the moment we proceed in an a-temporal setting.

Mean field theory, hereafter denoted MFT, replaces $\sum_{(i)} J(i-j)\omega_j$ by $E\{\sum_{(i)} J(i-j)\omega_j\} = m\sum_{(i)} J(i-j) \equiv mJ$ in expression (4.1.3) to obtain,

$$(4.1.3')\quad V_i(\omega_i, m) \equiv u(\omega_i) + Jm\omega_i + h\omega_i + \mu\varepsilon(\omega_i),\ \{\varepsilon(\omega_i)\}\ \text{IIDEV}.$$

Since $m \equiv \langle\omega_k\rangle$, mean field theory computes the average $\langle\omega_k\rangle'$ with respect to the

probabilities (4.1.3') and imposes the self consistency condition,

(4.1.4) $\langle\omega_k\rangle'=m.$

Equation (4.1.4) is a fixed point problem for m.

We shall see below how useful the MFT procedure can be to approximate quantities of interest. The procedure is much more general and can be carried out to higher levels of approximations in many different types of models. See Mezard etal. (1987), Ellis (1985), Kac (1968) and their references. However, the linkage of MFT and discrete choice theory presented below appears new to this paper.

Before we go further we wish to exhibit a connection between a Nash type notion of economic equilibrium in a Manski-McFadden world of interconnected discrete choosers and the MFT procedure.

Equation (4.1.3') leads to probabilities,

(4.1.5) $\hat{Pr}\{\omega\}\equiv\Pi\hat{Pr}\{\omega_i\},\ \hat{Pr}\{\omega_i\}=\exp\{\beta[u(\omega_i)+(Jm+h)\omega_i]\}/Z_i.$

Here $Z_i\equiv\sum\exp\{\beta[u(\nu_i)+(Jm+h)\nu_i]\}$, where $\sum$ is over $\nu_i\varepsilon\Omega_i\equiv\Omega$. Note that when $J=J_{ij}=0$, all i,j; then the probabilities given by (4.1.1) are identical to those given by (4.1.5). Also note, in our context, MFT may be viewed as the equilibrium generated by a group of individual agents i forming common expectations on the choice $\langle\omega_k\rangle=m$ of their neighbors, making their stochastic choices according to (4.1.3'), and having their expectations confirmed via the self consistency condition (4.1.4). We shall see examples below where the exact large system value of m is a solution to (4.1.4). The Kac method, which is exposited below, will give a theory of solution choice when there are multiple solutions to (4.1.4).

Consider the special case of (4.1.1) where $J_{ij}=J/N$. In this case the interaction strength goes to zero as $N\to\infty$, but every site has interaction strength J/N with every other site, no matter how distant. Hence, we have weak local interaction, but long range interaction.

For future use, for example, as inputs to formation of demand functions for risky assets, we want to find the limiting value of the following statistic: $\hat{m}\equiv M/N$, where $M\equiv\sum\omega_i$. The reader may wish to glance ahead at the next sections of the paper in order to see the key role that the "order parameter" m plays in the asset pricing models. We shall show,

(4.1.6) $\langle\hat{m}\rangle\to m^*,\ N\to\infty,$

Where m* solves (4.1.19a) below. Here "<.>" denotes expectation with respect to the probability (4.1.1) for the special case, $J_{ij}=J/N$. Details on how to define the object, <.>, will follow in due course. We show now, that the limiting value in (4.1.6) is given by a direct application of Kac (1968, p. 248).

In order to see how the Kac method works, let's do an example. A general treatment is in the Appendix. Put $J(i-j)\equiv J$ in (4.1.1). Let us compute $Pr\{\omega\}$, $Z\equiv Z_N$, and $\langle\omega_i\rangle$. We have

(4.1.7) $Z_N\equiv\sum\exp\beta[\sum u(\nu_i)+(J/2)(\sum\nu_i/N^{1/2})^2+h(\sum\nu_i)]P_N(\nu),\ \sum$ is over $\nu\varepsilon\Omega_N$.

Do the following steps. First, use the identity

(4.1.8) $\exp[a^2]=(1/(2\pi))^{1/2}\int\exp[-x^2/2 + 2^{1/2}xa]dx,$

and, second use the change of variable $y=x(\beta J/N)^{1/2}$ to obtain

(4.1.9) $\Pr\{\omega\}=(N/2\pi\beta J)^{1/2}\int\exp[-y^2N/2\beta J]\Pi\exp[\beta u(\omega_i)+(y+\beta h)\omega_i]dyP_N(\omega)/Z_N,$

(4.1.10) $Z_N\equiv(N/2\pi\beta J)^{1/2}\int\exp[-y^2N/2\beta J]\Pi M[(y+\beta h)]dy,$

(4.1.11) $M(z)\equiv\int_{\xi\varepsilon\Omega}\exp[z\xi+\beta u(\xi)]dF$, Π is product over $i=1,2,\ldots,N$.

Note that we use "M" to denote "moment generating function" for (4.1.11). Compute, observing that $\langle\omega_i\rangle=\langle\omega_j\rangle$ for all i, j,

(4.1.12) $m=\lim\{\langle[(1/N)(\sum\omega_i)]\rangle\}$

$$=\lim\{\int g(\beta h+y)[K(y)]^N dy/\int K(\vartheta)^N d\vartheta\}\equiv\lim\int g(\beta h+y)\mu_N(dy),$$

where, $\mu_N(dy)\equiv[K(y)]^N dy/\int K(\vartheta)^N d\vartheta \Longrightarrow \delta_{y^*}(dy)$, $N\to\infty$,

(4.1.13) $K(y)\equiv M(\beta h+y)\exp[-y^2/2\beta J],$

(4.1.14) $g(\beta h+y)\equiv\int\{\xi\exp[\xi(\beta h+y)+\beta u(\xi)]dF(\xi)\}/M(\beta h+y)=M'(\beta h+y)/M(\beta h+y).$

Apply Laplace's method (cf. Kac, (1968, p. 248), Ellis, (1985, pp. 38, 50, 51)) to see that, as $N\to\infty$, all probability mass is piled onto $y^*\equiv\mathrm{Argmax}\{M(\beta h+y)\exp[-y^2/2\beta J]\}$, i.e., $\mu_N(dy)\Longrightarrow\delta_{y^*}(dy)$, $N\to\infty$. Hence,

(4.1.15a) $y^*\equiv\mathrm{Argmax}\{M(\beta h+y)\exp[-y^2/2\beta J]\},$

(4.1.15b) y^* solves $\beta JM'(\beta h+y)/M(\beta h+y)=y$, and m^* is given by,

(4.1.15c) $m^*=M'(\beta h+y^*)/M(\beta h+y^*).$

Note three things. First, (4.1.15a) demands that y^* be chosen to be the solution of (4.1.15b) which, in the case $u(.)\equiv0$, has the same sign as h when h is not zero. Second, note that $\beta Jm^*=y^*$. Third, observe that

(4.1.15d) $\beta(\lim_{N\to\infty}E\{\max_\omega G(\omega)\}/N)=\lim_{N\to\infty}(\ln(Z_N)/N)$

$$=\max_y\ln\{\exp[-y^2/(2\beta J)]M(y+\beta h)\}=\max_m\ln\{\exp[-m^2\beta J/2]M(\beta Jm+\beta h)\},$$

hence, the Kac (1968) method of solution selection amounts to choosing the social optimum solution of the "Nash" condition (4.1.19b) below. The point is this: Minimizing free energy in the thermodynamic limit to find the ground states corresponds to maximizing expected social welfare in the large economy limit to find the socially optimal states.

Now Ellis (1985, p.38) shows, for the case $u(.)\equiv0$, $c(z)\equiv\log[M(z)]$ is convex in z. Replace the measure dF(x) by $\exp[\beta u(x)]dF(x)$, and follow Ellis (1985, p. 229) to show c(z) is convex for general u(.), β. Therefore $c'(z)=M'(z)/M(z)$ nondecreases in z. Make the modest additional assumption

that $c'(z)$ increases in z. Thus $c(..)$ is 1-1 and it follows that

(4.1.16) $m=c'(\beta Jm+\beta h)=M'(\beta Jm+\beta h)/M(\beta Jm+\beta h)\equiv\theta(m)$.

In order to study equations (4.1.15), (4.1.16) first look at the special case, $\Omega\equiv\{-1,+1\}$, $u(-1)=u(+1)=0$, $dF(a)\equiv(1/2)\sum\delta_a$, where δ_a puts mass one on $a=-1,+1$, mass zero elsewhere. We have, recalling the definitions of hyperbolic cosine, sine, and tangent,

(4.1.17) $M(z)=\cosh(z)$, $M'(z)=\sinh(z)$, $c'(z)=\tanh(z)$,

(4.1.18) $m=\tanh(\beta Jm+\beta h)$.

Equation (4.1.18) is Ellis's Curie-Weiss mean field equation (Ellis (1985, p. 180, p. 182)). Turn now to discussion of this key equation.

Following Ellis it is easy to graph (4.1.18) and show that for $h=0$, there is only one solution, $m=0$; but, two solutions, $m_-=-m_+$, appear as soon as βJ becomes greater than one. For h not zero, (4.1.15a) requires the one with the same sign as h be chosen. A "phase transition" or "spontaneous magnetisation" is said to appear when βJ becomes greater than one. Turn now to the general case which includes the case, $u(-1)<u(+1)$.

For this case we have, from (4.1.15), denoting the optimum m by m^*,

(4.1.19a) $m^*=\mathrm{Argmax}\{\int\exp[\xi(\beta h+\beta Jm)+\beta u(\xi)]dF(\xi)\exp[-\beta Jm^2/2]\}\equiv\mathrm{Argmax}\{\Upsilon(m)\}$.

Hence m^* solves,

(4.1.19b) $m=\theta(m)\equiv\int\xi\exp[\xi(\beta h+\beta Jm)+\beta u(\xi)]dF(\xi)/\int\exp[\xi(\beta h+\beta Jm)+\beta u(\xi)]dF(\xi)$,

but (4.1.19a) gives the selection rule for the solution of (4.1.19b).

We summarize the discussion to this point into

<u>Proposition</u> <u>4.1</u>: For the special case of (4.1.1) with $J_{ij}\equiv J/N$, $\langle(1/N)\sum\omega_i\rangle\text{-->}m^*$, $N\text{-->}\infty$, where m^* solves (4.1.19a). The solution set to the first order necessary conditions for a maximimum in (4.1.19a) is the same as the solution set to the MFT equations (4.1.3'), (4.1.4), (4.1.5). However the limiting behavior of (4.1.1) gives a selection rule (4.1.19a) whereas the MFT equations do not. □

We remark that the value of $\langle\omega_i\rangle$ w.r.t to the MFT probabilities is easy to calculate using the product structure of the MFT probabilities. The calculation is similar to, but simpler than, the one carried out above. Turn now to the possibility of phase transition for the economics case $h=0$, $u(-1)$ not equal to $u(+1)$.

The intuition of the analysis of (4.1.18) suggests β, J large should lead to abrupt changes in m^* if $du\equiv u(+1)-u(-1)$ changes sign. Let us study (4.1.19a) to investigate this possibility. Our approach adapts Pearce (1981, p. 312-313).

Put $k\equiv\beta h+\beta Jm$ and rewrite (4.1.19b) thus

(4.1.20) $\theta(m)\equiv\Phi(k)=\int\xi\exp[\xi k+\beta u(\xi)]dF(\xi)/\int\exp[\xi k+\beta u(\xi)]dF(\xi)$

$$=\{\exp[k+\beta du]-\exp[-k]\}/\{\exp[k+\beta du]+\exp[-k]\}.$$

$=\tanh[\beta(Jm+h')]$, $h'\equiv h+du/2$.

We shall do a fairly complete analysis for the case, $dF\equiv(1/2)(\delta_{-1}+\delta_{+1})$, and content ourselves with suggesting possible extensions for general h, dF. Note that the R.H.S. of (4.1.20) shows us that replacing h by h' reduces (4.1.20) to an application of (4.1.18).

It is now straightforward to use (4.1.20) to check the following: (i) When du=0, $\Phi(k)$ is given by (4.1.18), $\gamma\equiv\beta J>1$ implies there is a phase transition, i.e., a positive and a negative root to (4.1.18) with the root having the same sign as h chosen by (4.1.19a). (ii) For fixed du, $\Phi(k)\to+1$, $k\to\infty$; $\Phi(k)\to-1$, $k\to-\infty$.

(iii) For $\beta>0$, for du>0 (<0) but close enough to zero and $\gamma\equiv\beta J>1$, the function $\Upsilon(m)$ in (4.1.19a) has two local maxima and one local minimum. The positive (negative) one is the global maximum. The global maximum $m^*=m^*(h',\gamma)$ is discontinuous at h'=0 for $\gamma>1$. All solution arcs m(h') are anti-symmetric with the local minima rising in h' and the local maxima falling in h'. The local mimimum arc starting at h'=0 satisfies $m(0,\gamma)=0$ and decreases in h'.

Under regularity conditions the solution properties outlined above can be generalized to the case where $dF(y)\equiv f(y)dy$, $f(-y)=f(y)$. Let $du(x)\equiv u(x^*)-u(x)$, where $x^*\equiv\mathrm{Argmax}\{u(\omega)\}$. In this case, for $du\equiv0$, one can show $c'(-z)=-c'(z)$, $M'(0)=\int\xi dF=0$, $M''(0)=\int\xi^2 dF$, $c'(0)=M''(0)$, so for h=0 two solutions $m_-=-m_+$ appear for $\beta JM''(0)>1$, and m=0 is the solution for $\beta JM''(0)<1$. Some conditions are needed on F to make c'(z) display the qualitative properties of tanh(z) which were used above. We summarize:

<u>Proposition</u> <u>4.1.2</u>: For the case $dF\equiv(1/2)(\delta_{-1}+\delta_{+1})$ phase transition behavior will appear, i.e., the maximum of (4.1.19a) will change discontinuously from negative to positive as $du\equiv u(+1)-u(-1)$ changes sign from negative to positive provided that $\beta J>1$. Under regularity conditions this result can be generalized to general F(.). □

Let $\Rightarrow$, ($\Rightarrow^p$) denote convergence in distribution (in probability). For use in further sections we need $\hat{m}\Rightarrow m^*$, $N\to\infty$. A natural strategy is to use the large deviations approach of Ellis (1985, cf. his references to the joint work of Ellis and Newman), but u(.) causes obstacles to arise in rewriting $\Pr\{\hat{m}\varepsilon A\}$ as a function of $\hat{m}$ and using large deviations theory to obtain a law of large numbers (cf. for example Ellis (1985, p. 99)). This obstacle seems to be a problem even if m* is a unique global maximum of (1.19a) with locally strongly concave behavior near m* (cf. Ellis and Newman (1978b,d) cited in Ellis (1985, p. 342)). However, it is not difficult to obtain a law of large numbers,

<u>Proposition</u> <u>4.1.3</u>: Assume m* is a unique global maximum of (1.19a) with locally strongly concave behavior near m*. Then $\hat{m}\Rightarrow m^*$.

Proof: Let $\varepsilon>0$. We use Chebyshev's inequality to prove $\hat{m}\Rightarrow^p m^*$. By Lukacs (1975, p. 33, 37), $\hat{m}\Rightarrow^p m^*$ implies $\hat{m}\Rightarrow m^*$. By Chebyshev's inequality (Lukacs (1975, p. 9)),

(4.1.21) $\Pr\{|\hat{m}-m^*|\geq\varepsilon\}\leq\mathrm{Var}\{\hat{m}-m^*\}/\varepsilon^2$,

so it is sufficient to prove $\mathrm{Var}\{\hat{m}-m^*\}\to0$, $N\to\infty$. We must show, $\langle[(1/N)\sum\omega_i-m^*]^2\rangle\to0$, $N\to\infty$. It is sufficient to show $\langle\sum\sum\omega_i\omega_j\rangle/N^2\to m^{*2}$. Hence, it is sufficient to show $\langle\omega_i\omega_j\rangle\to m^{*2}$, $N\to\infty$, for i not equal to j.

Show this by arguing as in (4.1.12), (4.1.14), (4.1.15) to show $\langle\omega_i\omega_j\rangle \to g(\beta h+y^*)^2=m^{*2}$, $N\to\infty$. □

We have briefly sketched the theory we need and have done a fairly complete job for the two state case, $\Omega=\{-1,+1\}$. Turn now to a very brief sketch of the K-state case for K>2.

The issue concerns construction of an IPS structure that is flexible enough to yield a "landscape" that is tunable to each of K choices. To see the problem look at the expression (4.1.7) copied below,

$$(4.1.22)\quad Z_N \equiv \sum \exp\beta[\sum u(\nu_i)+(J/2)(\sum \nu_i/N^{1/2})^2+h(\sum \nu_i)]P_N(\nu),$$

and note how the convex term $(J/2)(\sum \nu_i/N^{1/2})^2$ rewards going to the extremes of the choice set, Ω. Hence, this convex term plays a key role in determining the limit via (4.1.19a), therefore placing more elements into Ω is not likely to give us the flexibility we desire even if we move the mass points of F around at will.

As a tentative proposal to be investigated in more detail in a future paper we encode each of the K elements into a "bit string" of ±1's. One can encode 2^L elements using bit strings of length L. Let $\omega_{i.} \equiv (\omega_{i1},\ldots,\omega_{iL})\varepsilon\{-1,+1\}^L$. Define $u:\{-1,+1\}^L \to \mathbb{R}$. Define $u\equiv-\infty$ for some bit strings in order to deal with cases where K is not equal to 2^L for some L. Define Z_N by replacing the term, $(J/2)(\sum \nu_i/N^{1/2})^2$ with $\sum(J_l/2)(\sum \nu_i^l/N^{1/2})^2$, where the sum runs from l=1,...,L. Proceed as in the case L=1 to develop the limit theory.

The solution theory presented above will be used in the applications below. The applications will induce dynamics on the solution for $m\equiv m(\beta J,\beta h;u(.))$ by inducing parsimoniously parameterized functional forms for u(.), J as a function of past information. This, in turn will give us flexible functional forms of dynamics on volume and stock returns, which will be one of our key applications.

4.2. APPLICATIONS

In this section we deliver on the promised applications of the mathematical technology in Section 4.1. But before we get into details of the examples we should be clear about the goals we wish them to serve.

In particular we want the models to have the potential to contribute to explanation of the following stylized facts laid out nicely by the paper of Haugen, Talmor, Torous (1991) (hereafter, "HTT"). (i) HTT (1991, p. 987) point out the finding of Roll, Schwert, Cutler, Poterba, Summers as well as their own work that it is difficult to relate "volatility, changes in volatility, and significant price movements to real economic events." (ii) HTT (1991, p. 985) find "A majority of our volatility changes cannot be associated with the release of significant information." (iii) In studying the reaction of returns to changes in volatility HTT (1991, p. 1001) find there is an asymmetry in the "reaction of prices and subsequent mean returns [which is] consistent with non-linear risk aversion."

(iv) HTT (1991, p. 1003) stress the result of Roll that "much of the variance in the equity return series may be related to either private information or occasional "frenzy" unrelated to concrete information." (v) HTT (1991, p. 1003) stress Schwert's finding: "Schwert (1989), in an exhaustive study, finds that the volatility of stock returns are not closely related to the volatility of other economic variables such as long and short term interest rates, the money supply, and inflation rates." (vi) HTT (1991, p. 1004) stress "..the fact that we find a highly significant, positive price reaction to volatility decreases...the fact that the price adjustments are followed by directionally consistent adjustments in mean realized returns...further reinforce our confidence that, on average over all events, we are seeing a reaction to changes in risk as opposed to expected cash flow." With this factual background in place let us return to the examples.

Examples 4.2.1-4.2.3 concern equilibrium asset pricing models where all traders have mean variance demands and some traders have biases in their expectations. Example 4.2.1 contains traders with biased expectations where IPS theory is used to parameterize interdependence across biases and to locate sufficient conditions for an effect of biased traders to remain in the large economy limit. The example suggests uses of IPS theory to parsimoniously parameterize interdependence of biased expectations in such a way that econometric techniques based upon orothgonality conditions may be used to estimate the parameters and test for the presence of biased traders. Examples 4.2.2 and 4.2.3 are variations on this theme.

Section 4.3 exposits an example which shows how interdependence across agents in correlations of their own-income with the market leads to an adjustment in conventional asset pricing formulae as well as a source of equilibrium trading volume. Section 4.4 contains a version of Campbell, Grossman, and Wang's (1991) model with interdependence in trader risk tolerances where the degree of interdependence may depend upon the past. The fifth section, 4.5, contains a version of Hellwig's (1980) model with inderdependence in signal quality. Stephen Durlauf has stressed the point that this kind of model can show how abrupt market movements can be caused by changes in the degree of correlation of information between agents rather than by large changes in information.

Section 4.6 briefly shows how interdependent firms in the Lucas (1978), Brock (1982) asset pricing models can lead to large movements in asset prices. The examples are all unified by showing how parameterization of the degree of interdependence by IPS modeling leads to analytically tractable equilibrium dynamics in the large economy limit which are suggestive of pathways through which small changes can have large impacts. Turn to describing demand functions.

The demand functions stress three channels of heterogeneity: (i) differing risk aversion parameters, (ii) differing expectations or beliefs, (iii) differing covariance structure of own-income with the marekt. Let trader i have demand

$$(4.2.1)\quad D_i(p)=\tau_i E_{it}q'/V_{it}(q')-Cov_{it}(q',w_i')/V_{it}(q'),$$

where p is asset price, τ is risk tolerance, E_{it}, V_{it}, Cov_{it} are conditional mean, variance, covariance on information available to i at date t, $q'\equiv p_{t+1}+y_{t+1}-Rp_t\equiv$excess return at t+1, p_{t+1}, y_{t+1} are asset price and asset dividend (or net cash flow) at date t+1, $R\equiv 1+r$ is return on a risk free asset, $w_i'\equiv w_{i,t+1}$ is other sources of income to i at date t+1. We shall often denote $x\equiv x_t$, $x'\equiv x_{t+1}$, for any quantity, x, to save typing.

The demand function (4.2.1) can be obtained from a two period overlapping generations setup where each trader gets first period income which is allocated between the risky asset and the risk free asset. Utility is obtained from consumption of all wealth in the second period. Wealth comes from (i) other sources of income, (ii) earnings on the two assets. The demand function (4.2.1) is derived by maximizing conditional expectation of mean-variance utility or, under normally distributed returns, by maximizing conditional expectation of exponential utility.

The assumption of two period lived traders is restrictive, but it should be clear that the methods laid out here can be generalized to handle traders with arbitrary lives.

In (4.2.1) there are three channels by which trader characteristics could be related: (i) expectational differences; (ii) risk tolerances; (iii) covariances of excess returns with own-income.

First we deal with E_{it}, V_{it}. Nelson (1992) has shown, in a diffusion context, that frequent sampling within a period can produce an estimator of the conditional variance that is much more precise than the best estimator of the conditional mean. For this reason and for simplicity we shall assume $V_{it}\equiv V_t$, is independent of i. It will be apparent as we illustrate our methods that this assumption can be relaxed at the cost of considerable complexity.

Assume, $w_i'=\rho_i(p'+y')+\varepsilon_i'$, and ε_i'conditionally independent of $p'+y'$, divide both sides of (4.2.1) by N, sum over i to obtain (conditional on the history of the economy at date t), suppressing "t" for ease of notation,

$$(4.2.2)\quad (1/N)\sum\{\tau_i E_i q'/V(q')-\rho_i\}$$

$$==>(1/V(q')[E^*\{\tau_i E_i q'\}]\}-E^*\rho_i,\ N-->\infty,$$

where E^* denotes expectation w.r.t to the measure, $\mu^*(A)$, defined by

$$(4.2.3)\quad (1/N)\sum 1[(\tau_i, I_i, \rho_i)\varepsilon A]==>\mu^*(A),\ N-->\infty.$$

Here I_i denotes the information set of trader i at date t, A is a set of agent characteristics (which includes choices), $1[(\tau_i, I_i, \rho_i)\varepsilon A]$ is the indicator function of the event $[(\tau_i, I_i, \rho_i)\varepsilon A]$ which is unity if $(\tau_i, I_i, \rho_i)\varepsilon A$, zero otherwise, and ==> denotes weak convergence. The theory of Section 4.1 locates sufficient conditions for the weak convergence of (4.2.3). We shall

assume without further mention that these sufficient conditions hold.

Suppose there are x shares outstanding per trader. Then equilibration of demand and supply per trader yields, in the large economy limit, by (4.2.2),

(4.2.4) $(1/V(q')[E^*\{\tau_i E_i q']\}-E^*\rho_i=x$

We show the value of the modelling of Section 1 by applying it to a sequence of examples based on the above.

EXAMPLE 4.2.1

Consider the "noise trader" theory of DeLong, Shleifer, Summers, and Waldman (1990), hereafter "DSSW." Let us use the theory of section 4.1 to locate sufficient conditions for noise trader risk to matter in the large economy limit and to suggest a method of estimating the effect of noise traders using the methodology of Hansen and Singleton (1982).

For simplicity assume homogeneous conditional expectations on variance and an estimation procedure for the conditional mean with the following structure of errors across the set of noise traders, $\Omega=\{\text{bear, bull}\}^2\equiv\{-1,+1\}^2$,

(4.2.5) $E_{it}(p'+y')=b_0\omega_{0it}+[1+b_1\omega_{1it}]E_t(p'+y')$,

where at each date t, $E_t(p'+y')$ is conditional expectation on a common information set available to all N traders, $\{\omega_{.it}\equiv(\omega_{0it},\omega_{1it})\}$ is distributed according to a product form (like Example 4.2.3 below) of (4.1.3') (4.1.5) where $u(\omega,t)$ is parameterized according to a measure of how well belief ω produced risk adjusted profits (utility) in the past.

In DSSW (1990) the bias in expectation is additive IID so $b_1=0$ captures the flavor of DSSW. So let us put $b_1=0$ for specificity. But, the reader should keep in mind that we can deal just as easily with multiplicative errors as additive errors. Put $x=\rho=0$, assume constant risk tolerance across agents, bring back subscripts for clarity, in (4.2.4) to obtain, from (4.1.19a,b),

(4.2.6) $Rp_t=b_0m_t^*+E_t(p_{t+1}+y_{t+1})]$.

Write (4.2.6) in the form

(4.2.7) $E_t\{b_0m_t^*+(p_{t+1}+y_{t+1})-Rp_t\}=0$.

Equation (4.2.7) can be used to generate a set of orthogonality restrictions so that the parameters b_0, and the parameters embedded in m_t^* via (4.1.19) may be estimated (given a specification of behavior of $(\beta,J,h,u(.))$ over time) following the Generalized Instrumental Variables (GIV) used by Hansen and Singleton (1982). We speculate that the parameters of rather elaborate dynamic specifications could be estimated by adapting the simulation estimator methods of Hotz, Miller, Sanders, and Smith (1992). In this way returns data can speak to testing for the presence of noise traders with, for example, additive errors in formation of conditional expectations by testing H_o: $b_0=0$ against the alternative H_a: b_0 not zero.

Of course somce conditions must be imposed for the GIV procedure to "identify" the parameters of interest. A more serious problem with testing (4.2.7) concerns confusion of movements of the marginal rate of substitution

in the CCAPM (Lucas (1978)) context tested by Hansen and Singleton with presence of noise traders in the context (4.2.7). But this problem could be dealt with by a noise trader component into the CCAPM setup of Hansen and Singleton (1982), following a procedure analogous to the above and deriving a general set of orthogonality conditions in which both the "pure" Hansen and Singleton CCAPM and the "pure" noise trader models are "nested."

Example 4.1 shows how a rich class of models may be formulated that (i) are econometrically tractible to GIV methodology, (ii) can be used to locate sufficient conditions for noise trader effects to survive the washing out effect of the law of large numbers, (There must be aggregate shocks to the u(.,t) or $\beta J>1$), (iii) can be used to locate sufficient conditions for the additive IID errors of DSSW (1990) to appear in the large economy limit, (iv) can be enriched by different parameterizations of the u(.,t) in (4.1.3'). We point out in passing that the presence of noise trading effects in the context (4.2.6) can be tested by using the West (1987) test. His procedure tests for the presence of terms like b_0 in linear present value models (4.2.6).

This is a good point to add a few words about justification for study of models with dispersion of beliefs. Antoniewicz (1992) in her work on volume reviews received work on volume dynamics. The consensus of this work is that trading volume is a very persistent series that is difficult to reduce to white noise by standard "detrending" methods.

Sargent (1992) shows how hard it is to preserve volume persistence in settings where the no-trade theorem becomes operative through learning. Therefore it appears that persistence in belief disparity will be needed if one is to get volume persistence out of belief disparity. While we shall exhibit models below that generate volume dynamics from heterogeneity in risk aversion and correlations of own income with the market these models do not seem right for explaining high frequency volume dynamics.

One justification for persistence in belief disparity is the work of Kurz (1990, 1991, 1992) who develops a theory where all traders see the same data, form bulk quantities such as time averages, all time averages converge for each trader, yet disparity in limiting quantities remain. There is enough stationarity in Kurz's setting so that time averages converge, yet there is enough nonstationarity that each agent may not converge onto the same probability (the true probability). For the context of persistence of belief disparity it may be useful to think of Kurz's setting as a metaphor for a situation where data is arriving fast enough for each individual trader's estimators using time averages to converge but where the underlying system dynamics is changing slowly but fast enough that traders do not "lock onto" common agreement about the underlying probability. I.e. their estimators do not converge onto common limits.

Our type of modeling may have use in the future as a way of locating sufficient conditions on the degree of dependence of individual beliefs so that an aggregative effect remains in the cross sectional large economy limit. Kurz (1992) uses his theory to argue that the Dow was grossly overvalued in 1966. This argument requires that belief bias remain in the large economy limit. It is beyond the scope of our paper to say more about Kurz's stimulating work here. Suffice it to say that we believe that belief disparity plays an important role in volume dynamics and study of such models is justified. The dynamics of such models may be usefully disciplined by evolutionary modeling as in Blume and Easley (1992). Turn now to a related class of examples.

EXAMPLE 4.2.2

Brock (1991a, p. 136-137) sketches a model where each trader has a choice

of two strategies: "-1" equals a chartist "trend chasing" strategy and "+1" equals a "fundamentalist" strategy. Each of these strategies is a recipe for updating their estimate $E_{it}(p'+y')$ at each date t. Traders keep a record of the profits earned by the two strategies. Brock (1991a, p. 136-137) updated the "field" parameter h_t in (4.1.7) as a function of relative profits at t. We improve on this by using the theory in Section 4.1.

It is more natural to put $h_t=0$ in (4.1.7) and define $u(\omega,t)$ to be the estimated profit for strategy $\omega\varepsilon\Omega\equiv\{-1,+1\}$, where the estimate is based upon the common information I_t available to traders at date t.

We define the fundamental strategy by putting $E_{+1,t}(p'+y')=E_t y'+E_t p'_{F,t+1}$ where $\{p_{Ft}\}$ is the forward rational expectations solution process of the equation $Rp_t=E\{p_{t+1}+y_{t+1}|I_t\}$. As in Brock (1991a, p. 136), for strategy $\omega=-1$, put, $q_t\equiv p_t+y_t$,

(4.2.8) $E_{-1,t}q_{t+1}\equiv p_{Ft}/R+b_t$, $b_t\equiv b_{t-1}+\lambda(q_{t-1}-MA(l,t-1))$,

(4.2.9) $MA(l,t-1)\equiv[q_{t-1}+\ldots+q_{t-1-(l-1)}]/l\equiv$moving average with l lags.

Suppose $\lambda>0$. Note that $q_{t-1}>MA(l,t-1)$ causes the bias over the fundamental to be increased; vice versa for "<."

Assume, for clarity that $\tau_i=\tau$, $Cov_{it}\equiv 0$, $x=0$. Close the model by using the expectation $E_{\omega_{it}}(q_{t+1})$, $\omega_{it}\varepsilon\{-1,+1\}$ to form the demands (4.2.1). Assume, at each date t, the probability trader i chooses ω_{it} is given by the MFT-discrete choice model (4.1.3'), (4.1.5).

We have a mixed discrete/continuous choice problem where (4.1.3') serves as the discrete choice model for which strategy (conditional expectation) to use in forming demands. The continuous choice problem is the choice of optimum quantity of stock and bond to purchase given the conditional expectation (strategy). For each fixed date t, the $N\to\infty$ equilibrium (4.2.4) may be rewritten

(4.2.10) $Rp_t=[(1-m^*_t)/2]E_{-1,t}(q_{t+1})+[(1+m^*_t)/2]E_{+1,t}(q_{t+1})$,

where we choose m^*_t to be the largest (in absolute value) solution with the same sign as du to,

(4.2.11) $m=\{\exp[\beta Jm+\beta du]-\exp[-\beta Jm]\}/\{\exp[\beta Jm+\beta du]+\exp[-\beta J]\}=\tanh[\beta(Jm+h')]$,

where $h'\equiv du/2$, $du\equiv u(+1,t)-u(-1,t)$, and $u(\omega,t)$ are measures of how well following strategy ω has generated utility for the trader had he followed it in the past. We assume this measure is a matter of public record available to all traders, but choice of ω is governed by (4.1.3'). One may now study the dynamics generated by (4.2.10). Unfortunately we must leave it to future research.

EXAMPLE 4.2.3 (Based on Arthur (1992))

Brian Arthur has written an interesting paper where he argues for

replacing the deductive mode of theorizing by an inductive mode of theorizing. He shows that inductive modes are analytically tractible by considering a stock market where traders take positions by monitoring a collection of predictors $H_1,\ldots,H_P$. Suppose we encode these using bit strings $\omega\varepsilon\{-1,+1\}^L$ of length L as suggested at the end of Section 4.1. Introduce social interaction terms for each slot of the bit string and introduce a record for each predictor on how well it has done in the past. Base the utility $u(\omega,t)$ on this record at t. Let, at each date t, discrete choice occur according to the natural generalization of the discrete choice model (4.1.3'). Then join Arthur's approach and Example 4.2.2 to develop the dynamics. Our modification of Arthur allows "herding" which is induced by the interaction terms $\{J_l\}$.

The dynamics of this modified Arthur model should be very rich. It would be interesting to simulate it and see how easy it is to find parameters such that the output of returns and volume replicate the stylized facts reported by HTT which were discussed above. In principle the parameters of this modified Arthur model could be fitted to a subset of data to replicate relevant moments in sample. Then it could be evaluated by tests out-of-sample. Turn now to an example that generates trading volume via heterogeneity in correlations of own income with the market portfolio.

SECTION 4.3: A MODEL WITH VOLUME AND PRICE DYNAMICS

The volume dynamics are complicated in the general model (4.2.4), but they can be worked out and volume data may be used in estimation. However, simple volume dynamics may be obtained from (4.2.4) with

$$(4.3.1)\quad w_i'=\rho_i(p'+y')+\varepsilon_i',$$

where $\{\rho_i\}$ has the probability structure (4.1.1), ε_i' is independent of p',y' and satisfies $(1/N)\sum\varepsilon_i'\Longrightarrow 0$.

Assume there is supply of x shares per trader. Assuming homogeneous expectations on conditional mean and variance in (4.2.4), equating demand to supply of shares for N traders yields, introducing a first type of trader which has all ρ_i equal to a constant, we have,

$$(4.3.2)\quad x=(1/N)\sum D_i(p)=(1/N)\sum[\tau E_t q'/V_t(q')]-\rho_i]=\tau E_t q'/V_t(q')-n_1\rho_1-n_2\hat{\rho}_2,$$

where

$$(4.3.3a)\quad \hat{\rho}_2\equiv\sum_i\rho_i/N_2\Longrightarrow\rho_2(m),\ N\dashrightarrow\infty,\ N_k=n_kN,\ n_k \text{ fixed},\ k=1,2.$$

Note that q' depends upon N but we abuse notation by neglecting this dependence in the notation. Here we suppose ρ_1 is constant across the N_1 type one traders, $\rho_2(\omega_{it})$ is the state of correlation for type two traders where $\Pr\{\omega\}$ is given by (4.1.5).

This raises an issue of interpretation. One interpretation is to put u(.)=0 and simply treat (4.1.5) as a convenient way to parsimoniously parameterize cross dependence of ρ in group two. Equation (4.1.5) may be motivated by placing the traders on a Durlauf (1991a,b) type lattice with probability structure (4.1.3) on the ρ's of (4.3.1). The lattice captures the relatedness of trader own incomes to each other. Equation (4.1.5) is an MFT approximation to (4.1.3) that is rough, but is accurate enough to suggest sufficient conditions for phase transition type behavior to take place (cf.

Pearce (1981)). In any event this parametrization forces one to realize that *some* measure of cross dependence plays a key role in preventing the law of large numbers from "washing out" the ρ-effect, i.e, preventing $\hat{\rho}_2$ from converging to 0, as N-->∞ unless this is "forced" by putting h not equal to zero. Small changes in h (or u(.)) can lead to large effects only when *some* measure of cross dependence is big enough. Equation (4.1.5) seems as attractive a way to capture this kind of effect as any.

Another interpretation is to imagine a discrete menu of funds with the same conditional variance but varying correlation with own income for group two traders. Consider the special case of a low correlation fund, "-1" and a high correlation fund "+1." Let a measure of past performance of each fund $u(\pm 1,t)$ be available at each date t. Then each member of group two picking which fund to buy shares in according to the discrete choice model (4.1.3) will lead to (4.1.5).

In this two state case, at each point in time, the limiting value of $\hat{\rho}_2$ will be

(4.3.3b) $\rho_2(m)\equiv[(1-m)/2]\rho_2(-1)+[(1+m)/2]\rho_2(+1).$

Solve (4.3.2) for $E_t q'/V_t(q')$ to obtain

(4.3.4) $E_t q'/V_t(q')=[x+n_1\rho_1+n_2\rho_2(m)]/\tau\equiv z_t.$

In order to simplify the volume dynamics we ignore trading within group two and measure trading *across* groups one and two. Denote by D_{kt} the equilibrium demand by trader group k=1, 2. With this qualification a natural measure of trading per capita per share can be generated from the following, which must hold in equilibrium,

(4.3.5) $D_{1t}-D_{1,t-1}=n_1\tau(z_t-z_{t-1})$

Motivated by (4.3.5) we define the turnover measure over the period [t-1,t], denote it by V_t,

(4.3.6) $V_t\equiv n_1\tau(z_t-z_{t-1})/x.$

Equation (4.3.6) can be turned into a useful equation by parameterizing the volume dynamics via parameterization of $\{u_t(.),J_t,h_t\}$ as functions of, for example, past y-innovations and past volume. Given a probability structure on $\{y_t\}$, for example, AutoRegressive with Independent and Identically Distributed (IID) or Martingale Difference Sequence (MDS) innovations, and a derived dynamics for $\{m_t\}$, where $m_t\equiv m(J_t,h_t;u(.))$; equation (4.3.4) may be solved by forward iteration. This can be written as the conditional expectation of a capitalized sum of "adjusted" earnings where the capitalization factor is 1/R. Both the price and volume dynamics can display abrupt changes to small changes in $u_t(.),h_t$ when $\beta J_t>1$. We believe it would be interesting to "calibrate" models like Examples 4.1-4.3 and see how many of the stylized facts listed by HTT can be replicated. More will be said about this and other applications below.

4.4: A RATIONAL EXPECTATIONS MODELS OF TRADING VOLUME AND LIQUIDITY PROVIDERS

Campbell, Grossman, and Wang (1991) have developed a rational expectations model with two types of traders. Type A have constant risk aversion parameter "a" and type B have stochastic risk aversion parameter "b_t" at time t. We use the probability structure of section 1 to "derive" a stochastic dynamics for b_t. We outline how the model may be "solved" for a closed form solution by a dynamic variational approximation analysis.

We use similar notation as CGW. Put $R\equiv 1+r$, $r>0$ equal to return on the risk-free asset which is in perfectly elastic supply. Let $\bar{X}$ be supply per capita of stock, each share pays $D_t=\bar{D}+d_t$, $d_t=\alpha d_{t-1}+u_t$, $0\le\alpha\le 1$, u_t $IID(0,\sigma_u^2)$. There are two type of investors A,B with mean variance demands,

$$(4.4.1)\quad X_t^k=E[Q_{t+1}|I_t]/\psi_k Var[Q_{t+1}|I_t],\ \psi_A\equiv a, \psi_B\equiv b_t,\ I_t=(P_t,D_t,S_t),$$

where $Q_{t+1}\equiv P_{t+1}+D_{t+1}-RP_t\equiv$excess returns, $u_{t+1}=S_t+\varepsilon_{t+1}$, $\{S_t,\varepsilon_{t+1}\}$ is jointly IID with both means zero, $E[u_{t+1}|S_t]=S_t$, $Var[u_{t+1}|S_t]=\sigma_\varepsilon^2$, $Var[S_t]=\sigma_s^2$. Put

$$(4.4.2)\quad Z_t\equiv ab_t/[(1-\omega)a+\omega b_t],\ \omega\equiv\text{fraction type A},$$

assume $\{Z_t\}$ satisfies $E[Z_{t+1}|Z_t]=\gamma_0+\gamma_1 Z_t$, $0\le\gamma_1\le 1$, $Var[Z_{t+1}|Z_t]=\sigma_Z^2$, assume $\sigma_Z^2\le(R-\gamma_1)^2(R-\alpha)^2/4\bar{X}^2(R^2\sigma_\varepsilon^2+\sigma_s^2)$. Then CGW show there is an equilibrium price function of the form,

$$(4.4.3)\quad P_t=p_0+p_1 d_t+p_2 Z_t+p_3 S_t,\ p_1,p_3>0,\ p_2<0,$$

$$(4.4.4)\quad p_1=\alpha/(R-\alpha),\ p_3=1/(R-\alpha),\ p_0=(1/(R-1))[\bar{D}+\gamma_0 p_2],$$

$$(4.4.5)\quad p_2=(1/(2\bar{X}\sigma_Z^2))\{-(R-\gamma_1)+[((R-\gamma_1)^2-4(1/(R-\alpha))^2(\bar{X}\sigma_Z^2)(R^2\sigma_\varepsilon^2+\sigma_s^2)]^{1/2}\},$$

$$(4.4.6)\quad Q_{t+1}=(\bar{D}-rp_0)+p_2[Z_{t+1}-RZ_t]+(1/(R-\alpha))S_{t+1}+(R/(R-\alpha))\varepsilon_{t+1}.$$

Add the demands, use the market clearing condition and the form of the solution price function to obtain

$$(4.4.7)\quad E[Q_{t+1}|I_t]=(\bar{X}\sigma_Q^2)Z_t,\ Var(Q_{t+1}|I_t)\equiv\sigma_Q^2=(1+p_1)^2\sigma_\varepsilon^2+p_2^2\sigma_Z^2+p_3^2\sigma_s^2.$$

Note that (4.4.7) says that excess returns are positive with the size increasing as the measure of average risk aversion, Z_t increases. Excess returns also increase as the conditional variance increases. However, note that conditional variance is constant. Hence the CGW model is not able to explain the well known serial correlation structure of conditional variance, i.e. the AutoRegressive Conditional Heteroscedasticity (ARCH) documented by the studies cited by Bollerslev, Chou, and Kroner (1992). This is because the CGW model is a linear model. Turn now to a nonlinear model which nests the CGW model.

Let there be three types of investors, A,B,C. Types A,B are as in CGW. At date t, member i of type C has risk tolerance given by $T_c(\omega_{it})$. Passing to the limit as the number of traders, N, goes to infinity but holding the fractions n_k, k=A,B,C fixed we have, equating demand to supply,

(4.4.8) $E[Q_{t+1}|I_t]=(\bar{X}\sigma_{Qt}^2)Z_t$, $\sigma_{Qt}^2 \equiv Var(Q_{t+1}|I_t)$, $Z_t \equiv 1/[n_a\tau_a+n_b\tau_{bt}+n_c\tau_c(m_t)]$,

where $\tau_c(m)=[(1-m)/2]T_c(-1)+[(1+m)/2)]T_c(+1)$, $m_t=m(J_t,h_t)$, $\tau_{bt}=1/b_t$, $b_t \equiv$risk aversion of type B as in CGW. If $\{(J_t,h_t)\}$ is a stochastic process such that $\{Z_t\}$ satisfied $E[Z_{t+1}|Z_t]=\gamma_0+\gamma_1 Z_t$, $0\le\gamma_1\le 1$, $Var[Z_{t+1}|Z_t]=\sigma_Z^2$ we could simply copy CGW and find their equilibrium price function.

But we want to parameterize $\{(J_t,h_t;u(.))\}$ as a function of past volume and past returns in such a way that we have the potential to replicate the stylized facts collected by HTT. This requires a nonconstant σ_Z^2 and a natural way to introduce this is to parameterize J_t, h_t as functions of the past. For example, a large "aggregate dividend surprise," $D_t-E_{t-1}D_t$, may be associated with a change in the degree of dependence of risk tolerances in the future, i.e., a change in J_{t+1}.

While it is beyond the scope of this article to develop them, there are two routes to dealing with the third class of traders in the CGW model. The first one is to take a parameter like n_c and expand the equilibrium in a Taylor series in n_c around the value $n_c=0$. In this way one can exploit the known CGW solution ($n_c=0$) to build up an approximation to the unknown solution for positive n_c. The second route is to solve T period problems by backwards "dynamic programming" from a known terminal value p_T at T. A typical value for p_T is zero.

4.5: AN ASYMMETRIC INFORMATION RATIONAL EXPECTATIONS MODEL

Hellwig (1980) is a well known paper that derives a closed form solution for the large economy limit for a rational expectations model where N traders each receive signals about the future earnings of an asset. The solution shows how information is aggregated by the rational expectations price function in a competitive market.

Fix date t, suppress "t" in the notation, and append to Hellwig's model the following probability structure of signal quality across the set of N traders. If trader i is in state -1, let her signal variance be $S^2>s^2$ which is her signal variance in state +1. Let $\omega\equiv(\omega_1,\ldots,\omega_N)$, $\omega_i\varepsilon\{-1,+1\}$ denote a configuration and let configuration probabilities be given by the Curie-Weiss probabilities treated in (4.1.17), (4.1.18) above. We have positioned ourselves to use Section 5 of Hellwig (1980) where he derives the form of the equilibrium price function in the large economy limit.

Define a trader to be "informed" if she is in state +1 so that her signal variance, s^2 is small. Traders in state "-1" are "uninformed." Now check that Hellwig's Assumptions B.1-B.4 are satisfied and take the large economy limit. Assume $(X,Z,\varepsilon_1,\ldots,\varepsilon_N)$ is Gaussian conditional on ω with the same diagonal variance covariance structure as Hellwig. Let f_-, f_+ denote the limiting fractions of uninformed and informed traders.

Look at Hellwig's equations (1980, p. 492), where we use his notation except we suppress the "upper *," write random variables as caps, put A equal to risk tolerance, and $B=A[f_-/S^2+f_+/s^2]$, where, by (4.1.17), (4.1.18),

(4.5.1) $f_-=(1-m)/2$, $f_+=(1+m)/2$, $m=\theta(m)=\tanh(\beta Jm+\beta h)$, for $u(.)$=constant,

(4.5.2) $P=\pi_0+\pi X-\gamma Z$,

(4.5.3) $\pi_0\equiv[\bar{X}\Delta^2A+\sigma^2\bar{Z}AB]/D$,

(4.5.4) $\pi\equiv[\sigma^2B\Delta^2+\sigma^2AB^2]/D$,

(4.5.5) $\gamma\equiv[\sigma^2\Delta^2+\sigma^2AB]/D$

(4.5.6) $D\equiv\sigma^2B\Delta^2+\sigma^2AB^2+A\Delta^2$

Concentrate first on the case $u(.)$=constant. If the mean field equation ,

(4.5.7) $m=\tanh(\beta Jm+\beta h)$,

has two solutions, choose the one with the same sign as h to be compatible with (4.1.19a).

The following four points may be made about this version of Hellwig's model. First, the correlatedness of the trader signal quality states may lead to a "phase transition" where the equalibrium price relationship makes an abrupt shift in reponse to small changes in (J,h). Stephen Durlauf has made the important point that this kind of model can be used to show how large market movements may be caused by changes in the degree of correlation of information between agents rather than by large changes in the information itself.

Second, the model raises issues of how to measure factors that might effect the correlation strength of signal quality across agents. This in turn impacts on how rapidly the price function impounds information and impacts on the likelihood of abrupt changes in returns which may appear to be blowoffs and crashes.

Gennotte and Leland (1990) study how the sensitivity of demand of each trader type demands upon relative quality of signals and how this feeds into above changes in the price relationship provided their outside hedging function is upward sloping. The formula above shows how similar behavior can be obtained without the need for such an outside hedging function. Also note that it may be possible to "endogenize" the outside supply of shares, Z, by a community of noise traders modelled as in Section two above. A generalization of Hellwig (1980) to allow a probability structure on signals themselves, rather than just signal variances, like that in Section 1 would allow more abrupt changes in the level of prices to a small amount of "news", but that attempt must await future research.

Third, note the qualitative role of the correlation structure of signal receipts of inducing abrupt changes in the equilibrium price function, and, hence, in equilibrium returns. This feature is likely to remain in more elaborate models.

A fourth point is this. We may introduce a discrete choice decision into the model where we allow agents to choose high signal quality strategy, "ω=+1," (for which a fee of F is paid each period) or choose low signal quality strategy "ω=-1,"(which is free). At each date t, choice is conducted according to the discrete choice model (4.1.3') where $u(\omega,t)$ is based upon a measure of past performance of strategy choice $\omega\varepsilon\{-1,+1\}$. Two separate cases can be treated: (i) $u(\omega,t)$ is updated according to a publically kept record of experience with strategy ω; (ii) $u(\omega,t)$ is updated according to each individual trader's experience with ω. Discrete choice model (4.1.3),

(4.1.3') governs the probability structure in both cases.

A version of this model under research parameterizes correlation strength J as a function of past volume and past "surprises" at the time slot frequency. This is an attempt to capture the idea that high information channel congestion forces traders to condition on "coarse" information sets such as past prices which should lead to higher J which leads to higher volatility, i.e., larger changes in response to vibrations in h, u(.). During periods of low congestion traders should be able to get better quality signals on X from more independent sources so that J should be lower. Regardless of the loose heuristics, the idea is to parameterize J,h, u(.) as functions of past price behavior, past volume, and past "surprises" (a measure of modulus of past forecast errors) in such a way that the data can speak to the form of this relationship. One version of this model that we have formulated leads to unpredictable first conditional moments of returns but somewhat predictable higher order conditional moments of returns.

The six applications above have been to financial models. We hasten to caution the reader that two period models and incomplete markets models, which we use to illustrate the usefulness of IPS methods are dangerous to apply in practice. This is partly because we have arbitrarily assumed that markets are incomplete in the Arrow-Debreu sense without giving a theory of why these markets are missing.

We have said nothing about the potentiality of options markets and other derivative security markets to ameliorate the potentiality for abrupt changes in returns in response to small events. Longer horizon models typically will lead to more smoothing behavior. More realistic models than those treated above will need to be investigated before it can be claimed that anything said in this paper pertains to financial reality. The point made in the financial section of this paper is simple: Models of this type are tractable to econometric methods such as Hansen and Singleton (1982), and Hotz etal. (1992). Indeed Tsibouris (1992) has estimated a version of an IPS model and tested the orthogonality restrictions with a degree of success comparable to received CCAPM theory. IPS models like those sketched above have the potential to help shed light on the puzzling stylized facts of HTT. Turn now to a very brief sketch how MFT/IPS/discrete choice methods may be useful in generating a new class of closed form solutions for simple macro/finance models.

4.6: A MACRO-FINANCE EQUILIBRIUM ASSET PRICING MODEL WITH INTERACTING AGENTS

We show off the flexibility of the approach to interactive systems modeling advertised above by exhibiting a macro-finance asset pricing model with a closed form solution. Consider Brock (1982, Example 1.5) where a representative "standin" consumer solves

(4.6.1) Maximize $E_0\{\sum_{t=0}^{t=\infty} \beta^{t-1}\log(c(t))\}$ s.t. $c_t+x_t=y_t\equiv\sum A_{it}x_{it-1}^{\alpha}$, $\sum x_{it-1}\leq x_{t-1}$,

where $c_t, x_t, x_{it}, A_{it}, y_t, \beta, \alpha$, denote consumption, capital stock, capital stock allocated to process i, productivity shock to process i, total output plus total capital stock carryover (all at date t), discount factor on future utility, and elasticity of production function. It is easy to see that the optimal solution of (4.6.1) is $x_t=\alpha\beta y_t$, $c_t=(1-\alpha\beta)y_t$, $x_{it}=\eta_{it}x_t$, where the $\{\eta_{it}\}$ solve

(4.6.2) Max $E_t\log(\sum A_{i,t+1}\eta_{it}^{\alpha}x_t^{\alpha})$, s.t. $\sum\eta_{it}=1$.

Note that (4.6.2) implies the $\{\eta_{it}\}$ do not depend upon x_t. We have now laid the foundation for building and solving an interacting systems model.

First, note that the solution form $x_t=\alpha\beta y_t$ does *not depend* upon the dynamic structure of $\{A_{it}\}$, hence we may preserve the same form of solution by introducing any pattern of externalities we wish and any number of agents we wish, so long as all of them are log utility maximizers facing problems with the same structure as (4.6.1), and all of them face the externalities parametrically when they solve their optimization problems. However, we wish to be able to compute statistics from aggregate quantities in order to make contact with Durlauf's (1991a,b) work on disparities among income and wealth across sites.

The solution for the $\{\eta_{it}\}$ in (4.6.2) is easy to find under the assumption that $Pr\{\omega_t\}$ is invariant to permutations within ω_t for each t. In this case we have

(4.6.3) $\eta_{it}=1/N$, for all i,t,

(4.6.4) $x_t=\alpha\beta\sum A_{it}(1/N)^{\alpha}x_{t-1}^{\alpha}$.

Given (4.6.3), (4.6.4) there are now two routes to obtaining a class of closed form solutions in the large economy limit, $N\to\infty$. First note that Section 1 locates sufficient conditions on the MFT/IPS probability structure for,

(4.6.5) $\sum A_{it}(1/N)\Longrightarrow E^{*}A_{it}$, $N\to\infty$,

so there is no problem for $\alpha=1$. Second, in order to deal with $\alpha<1$, consider an economy where $A_{it}=N^{1-\alpha}A_{0it}$. With this scaling (4.6.4) reduces to

(4.6.6) $x_t=\alpha\beta\sum A_{0it}(1/N)x_{t-1}^{\alpha}$.

One may now investigate asset prices following Brock (1982) for specific examples such as simple MFT parameterizations of $A_{it}\equiv A(\omega_{it})$ with $\omega=-1$ for low A, $\omega=+1$ for high A using the simple equations (4.1.17), (4.1.18). In this way one can show how $\beta J>1$, and an IID process for $\{h_t\}$ with mean zero and small variance can lead to big macro economic flucations.

A closely related type of example would be to replace the probability structure in Durlauf (1991a,b) with one of the MFT/IPS probability structures treated in this paper. The "Curie/Weiss" structure leading to (4.1.18) is simple enough to generate closed form solutions. The version of the discrete choice model reported by Proposition 4.1.2 is simple enough to apply to Durlauf's firms' choice of two technologies. While the resulting model would give something closer to a "closed form" solution, we doubt that it would be as rich as Durlauf's model.

SUMMARY, FURTHER REMARKS, AND CONCLUSIONS

This paper has tried to illustrate the usefulness of MFT/IPS methods as an input module into producing econometrically and analytically tractible models of use to finance and macroeconomics. We concentrated on finance and

stressed the potentially of MFT/IPS models of addressing stylized facts which stress the apparent lack of connection of movement of stock returns and volume to "fundamentals." This is a natural place to argue for the promise of this type of model in being able to deal with stylized facts such as HTT (1991).

HTT (1991, p. 1006) state: "The large number of volatility shifts that we detect, and the fact that we are unable to find significant, real economic events in the neighborhood of a majority of these shifts, lead us to the conclusion that we may be observing instability in the noise component of volatility stemming from the microstructure of the stock market. Thus while our findings support the notion that changes in risk premia may serve to partially explain the excess volatility observed in stock prices, the apparently excessive volatility of volatility which we observe only serves to raise further questions regarding our ability to account fully for the behavior of stock prices through current financial markets paradigms."

Note that HTT stress the lack of a linkage between real economic events and the volatility shifts, and the asset pricing models sketched above generate large changes in response to small changes in du or h provided $\beta J>1$. The parameter β is easy to interpret in the models built on the foundation of discrete choice such as (4.1.3). It is simply the intensity of choice and is a measure of the level of sharpness in choice. The parameter J is a measure of the strength of "ties" to a relevant "reference group" for each agent. Note that if intensity of choice is high we do not need much "sociology" for βJ to be greater than one. It is also plausible to think of parameterizing β, J as functions of the past history of the economy and estimating the parameters using, for example, the Generalized Instrumental Variable procedure (Hansen and Singleton (1982)). This is a good time to address a side issue that arises in IPS modeling.

IPS modeling is sometimes criticized in economics because it is said that there is no natural interpretation of the "inverse temperature" parameter β and even if there were the inverse temperature β is set exogenously such as controlling in a laboratory experiment or controlling by outside cooling or heating. Per Bak etal. (1992a,b) argue that sandpile models are superior to IPS models because the move to criticality is "self-organizing" rather than being forced exogenously.

While this argument has merit we believe that both types of models should be studied for the following reasons. (i) When IPS models are given a foundation in discrete choice random utility theory the interpretation of β becomes natural and we can imagine parameterizing it to capture economic incentives to make sharp or loose choices. (ii) The parameters J_{ij} become a tractable way to capture strong and weak ties between agents.

(iii) Since discrete choice econometric theory and IPS theory are well established we can draw on it to generate broad classes of econometrically tractable models as illustrated by the six examples above. Furthermore Anderson, de Palma, and Thisse (1993), show how there is a parallel between CES production functions and discrete choice theory and, hence, β is related to the elasticity of substitution in their CES production function. They show how welfare measures in discrete choice theory relate to production functions. The welfare measures treated in discrete choice theory are essentially the same as free energy expressions in IPS theory. This parallelism between economically interpretable quantities and physically interpretable quantities is beautiful and useful. (iv) Sandpile-based models still need an outside source (e.g. falling sand) to drive the pile to criticality. (v) The sandpile theory is not yet developed enough to conduct estimation and hypothesis testing which is fairly straightforward to do in the six examples laid out above. We conclude that it is wise to pursue both approaches because there are advantages and disadvantages to each.

APPENDIX

1. GENERAL PROBABILITY STRUCTURE WITH K TYPES OF INTERACTING AGENTS

The interactions will be considered over disjoint sets $A_1,\ldots,A_K$ where types are homogeneous within each set but heterogeneous across each set. The large system limit (as N≡total number-->∞) will be taken by holding the fraction of each type k=1,2,...,K constant. To formalize this let Ω be a set of real numbers, let Ω_N be its N-fold Cartesian product, $\omega\varepsilon\Omega_N$,

(1) $\Pr\{\omega\}=\exp[\beta G]P_N(\omega)/Z,\ G\equiv(1/2)\sum\sum M_k J_{kl}(N)M_l+\sum h_k M_k,$

where $M_k\equiv\sum\omega_i$ where $\sum$ is over i in A_k and $Z\equiv\sum\exp[G(\nu)]P_N(\nu)$ over all ν. Here $P_N(\nu)$ denotes the product probability on Ω_N induced by the common distribution function F on Ω. We will concentrate on the case where Ω is finite and F is a sum of "dirac deltas" but use $\sum$ and $\int$ interchangeably to suggest the natural extension to a continuous state space. We shall also assume the utility functions u(.) treated in Section 4 are constant. Once one sees how to generalize Section 4 for this case it will be straightforward to do it for utility functions.

The best way to think about this structure is to partition the vector ω thus: List first the components i in A_1, second the components i in A_2, etc. The probability structure captures homogeneous interactions within each set of entities $i\varepsilon A_k$ and captures heterogeneous interactions among entities across sets $A_1,\ldots,A_K$. The strength of interactions within A_k (across A_k,A_l) is measured by $J_{kk}(N)$ (by $J_{kl}(N)$) where the interaction strength will decrease linearly with N in this paper. That is to say the interaction strength becomes uniformly weaker across and within all sets of entities as N increases.

For future use, we want to find limiting values of the following statistics:

(2) $\hat{m}_k\equiv M_k/N_k \Longrightarrow \langle\omega_i\rangle,\ i\varepsilon A_k$, where

$N_k\equiv\#$ of elements of A_k, $N_k/N=n_k$, and, N_k, N-->∞ with n_k fixed. Here "<.>" denotes expectation with respect to the limiting probability, as N-->∞, defined by (1) and ==> denotes convergence in distribution. Details on how to define the object, <.>, will follow in due course. We show now, that if we put $J_{kl}(N)=I_{kl}/N$, I_{kl} constant, the limiting value of (2) is given by a small generalization of Kac (1968).

At the risk of repeating material in the text, in order to see the Kac method with a minimum of clutter, deal first with the case K=1, $I_{kl}\equiv J$, h(A)=h, $N_k=N$, $\hat{m}_k=\hat{m}$. Compute $\Pr\{\omega\}$, $Z\equiv Z_N$. We have

(3) $Z_N\equiv\sum\exp\beta[(J/2)(\sum\nu_i/N^{1/2})^2+h(\sum\nu_i)]P_N(\nu)$, $\sum$ is over $\nu\varepsilon\Omega_N$. Do the following steps. Put β=1 to ease notation. First, use the identity

(4) $\exp[a^2]=(1/(2\pi))^{1/2}\int\exp[-x^2/2 + 2^{1/2}xa]dx,$

and, second use the change of variable $y=x(J/N)^{1/2}$ to obtain

(5) $\Pr\{\omega\}=(N/2\pi J)^{1/2}\int\exp[-y^2N/2J]\Pi\exp[(y+h)\omega_i]dyP_N(\omega)/Z_N$,

(6) $Z_N\equiv(N/2\pi J)^{1/2}\int\exp[-y^2N/2J]\Pi M[(y+h)]dy$,

(7) $M(z)\equiv\sum_{\xi\varepsilon\Omega}\exp[z\xi]dF$, Π is product over $i=1,2,\ldots,N$.

Note that we use "M" to denote "moment generating function" for (7). Compute

(8) $m=\lim\{<[(1/N)(\sum\omega_i)]>\}=\lim\{\int g(h+y)[K(y)]^N dy/\int K(\vartheta)^N d\vartheta\}\equiv\int g(h+y)\mu_N(dy)$,

where, $\mu_N(dy)==>\delta_{y*}(dy)$, $N-->\infty$,

(9) $K(y)\equiv M(h+y)\exp[-y^2/2J]$,

(10) $g(h+y)\equiv\int\{\xi\exp[\xi(h+y)dF(\xi)\}/M(h+y)=M'(h+y)/M(h+y)$.

Apply Laplace's method (cf. Ellis (1985)) to see that, as $N-->\infty$, all probability mass is piled onto $y^*\equiv\mathrm{Argmax}\{M(h+y)\exp[-y^2/J\}$, i.e., $\mu_N(dy)==>\delta_{y*}(dy)$, $N-->\infty$. Hence,

(11) y^* solves $JM'(h+y)/M(h+y)=y$, $m=M'(h+y^*)/M(h+y^*)$.

Now Ellis (1985, p.38) shows $c(z)\equiv\log[M(z)]$ is convex, therefore $c'(z)=M'(z)/M(z)$ nondecreases in z. Make the modest additional assumption that $c'(z)$ increases in z. Then it is 1-1 and it follows that

(12) $m=c'(Jm+h)=M'(Jm+h)/M(Jm+h)$

In order to study equations (11), (12) look at the special case, $\Omega\equiv\{-1,+1\}$, $dF(a)\equiv(1/2)\sum\delta_a$, where δ_a puts mass one on $a=-1,+1$, mass zero elsewhere. We have, recalling the definitions of hyperbolic cosine, sine, and tangent,

(13) $M(z)=\cosh(z)$, $M'(z)=\sinh(z)$, $c'(z)=\tanh(z)$,

(14) $m=\tanh(Jm+h)$.

Equation (14) is Ellis's Curie-Weiss mean field equation (Ellis (1985, p. 180, p. 182) where we absorbed his β into J, h. Turn now to discussion of this key equation.

Following Ellis it is easy to graph (14) and show that for h=0, there is only one solution, m=0; but, two solutions, $m_-=-m_+$, appear as soon as J becomes greater than one. For h not zero the one with the same sign as h is chosen. A "phase transition" or "spontaneous magnetisation" is said to appear when J becomes greater than one.

Before turning to central limit theorems, we remark that the solution properties outlined above can be generalized to the case where $dF(y)\equiv f(y)dy$, $f(-y)=f(y)$ and some regularity conditions. In this case one can show $c'(-z)=-c'(z)$, $M'(0)=\int\xi dF=0$, $M''(0)=\int\xi^2 dF$, $c'(0)=M''(0)$, so for h=0 two

solutions $m_-=-m_+$ appear for $JM''(0)>1$, and $m=0$ is the solution for $JM''(0)<1$. Some conditions are needed on F to make $c'(z)$ display the qualitative properties of tanh(z) which were used above.

Ellis (1985, p. 187, p. 207, and reference to work of Ellis and Newman for general J, and h not zero) gives central limit theorems. In particular, for the case $J<1$, $h=0$ we have the central limit theorem

$$(15)\quad N^{1/2}(\hat{m}-m)\text{-->}N(0,\sigma^2(J,0)),\ N\text{-->}\infty,\ \sigma^2(J,0)\equiv(1-J)^{-1}.$$

Note how the variance tends to infinity as J tends to 1 from below.

<u>Remark</u>: It is easy to show using the same type of argument as that above that the covariances $<(\omega_i-m)(\omega_{i+L}-m)>=0$ in the limit for all integers L. That is why there are no covariance terms in (15). This appears to be a contradiction to the whole theme of this paper which is to show how models with correlated characteristics could be parsimoniously parameterized in such a way that econometric estimation is possible.

In order to explain this apparent contradiction we point out that Kac (1968, p. 258) shows that the Curie-Weiss probability structure we are using here is the limit as γ-->0 of a class of structures indexed by γ which contain local interactions which *do give nonzero correlations*. As γ-->0 the *range of interactions* becomes longer while the strength decreases in such a way that the Curie-Weiss equation (14) is obtained in the limit. In view of this "Kac bridge" between models with local strong interactions that have nonzero local correlations *whose strength increases with J* and the Curie-Weiss models with long range weak interactions that give the *same* equation (14) for the long run value of $<\omega_i>$ we shall speak of an increase in J as an increase of local correlation of characteristics. Kac (1968) develops a series of expansions in γ for solutions for his general model where the Curie-Weiss theory appears as the lowest order of accuracy but accurate enough to display the phase tranistion behavior that appears in the general model. In our view the analytical advantage of the Curie-Weiss structure and the Kac Bridge justifies the abuse of language we use in associating an increase in J with an increase in correlations across characteristics.□

Turn now to the general case. We shall use an identity exploited by Kac (1968). In the applications below, inducing dynamics will give us flexible functional forms of dynamics on volume and stock returns, which will be one of our key applications. Another key application will be dynamics of K macro aggregates.

GENERAL CASE: $K>1$.

Rewrite (1) as follows

$$(16)\quad Pr\{\omega\}=\exp[G]P_N(\omega)/Z,\ G\equiv(1/2)\sum\sum M_kJ_{kl}(N)M_l+\sum h_kM_k.$$

Put $N_j=n_jN$, $n_k^{1/2}J_{kl}(N)n_l^{1/2}\equiv n_k^{1/2}I_{kl}n_l^{1/2}/N\equiv J_{kl}/N$, J_{kl} constant,

$$(17)\quad G(\omega)\equiv(1/2)\sum\sum(M_k/N_k^{1/2})J_{kl}(M_l/N_l^{1/2})+\sum h_kM_k.$$

Following Kac (1968, p. 254) use the following identity,

(18) $\exp\{(1/2)\sum\sum\xi_i A_{ij}\xi_j\}=(2\pi)^{-K/2}[\det(\mathbf{A})]^{-1/2}\int\exp[\sum\xi_i x_i-(1/2)\mathbf{x}\mathbf{A}^{-1}\mathbf{x}]d\mathbf{x}$,

where $\sum$ is from 1 to K, bold face letters are vectors and matrices, $\int$ is over the K-vector $\mathbf{x}$, $\mathbf{A}$ is KxK.

Put $\mathbf{A}=\mathbf{J}$, $C\equiv(2\pi)^{-K/2}[\det(\mathbf{A})]^{-1/2}$, and write

(19) $\Pr\{\omega\}=CN[\Pi n_j]^{1/2}\int\exp[\sum M_k(h_k+z_k)-(N/2)\sum\sum B_{kl}n_k n_l z_k z_l dz/Z,\ z_j\equiv y_j/N_j^{1/2}$,

after making a change of variable from $\mathbf{y}$ to $\mathbf{z}$, letting the product Π run from 1,2,...,K, and putting $\mathbf{B}\equiv\mathbf{J}^{-1}$. Application of Kac's identity and summing term by term allows one to show that Z is given by

(20) $Z=CN[\Pi n_j]^{1/2}\int\{\Pi M(h_k+z_k)^{n_k}\exp[-(1/2)\sum\sum B_{kl}n_k n_l z_k z_l]\}^N dz$.

We are now in a position to compute the limiting values as N-->∞ of moments. Consider

(21) $\langle M_j/N_j\rangle$ for set A_j.

Compute, using (19), (20) to obtain

(22) $\langle M_j/N_j\rangle=\int[A(h_j+z_j)/M(h_j+z_j)]\{\Pi M(h_k+z_k)^{n_k}\exp[-(1/2)\sum\sum B_{kl}n_k n_l z_k z_l]\}^N dz/Z$.

Here $A(y)\equiv\int\{\xi\exp[\xi y dF(\xi)\}=M'(y)$. Use Laplace's method (Ellis (1985)), to observe that as N-->∞, all probability mass piles onto $\mathbf{z}^*$ where $\mathbf{z}^*$ maximizes,

(23) $\sum n_k\log M(h_k+z_k)-(1/2)\sum\sum B_{kl}n_k n_l z_k z_l$.

The first order necessary conditions for a maximum of (23) are given by

(24) $M'_k/M_k=\sum B_{kl}n_l z_l$, $M_k\equiv M(h_k+z_k)$.

Put $a_k\equiv M'_k/M_k$, $\mathbf{a}\equiv(a_1,\ldots,a_K)$, $c_k\equiv n_k z_k$, $\mathbf{c}\equiv(c_1,\ldots,c_K)$, and rewrite (24) thus,

(25) $\mathbf{a}=\mathbf{B}\mathbf{c}$, $\mathbf{J}\mathbf{a}=\mathbf{c}$.

Recall that $J_{kl}=[n_k n_l]^{1/2}I_{kl}$ so (25) becomes

(26) $\sum_k[n_k n_l]^{1/2}I_{kl}\ M'_k/M_k=n_l z_l$, $l=1,2,\ldots,K$.

Note, in the diagonal case, $I_{kl}=0$, for k not equal to l, that n_l cancels from both sides of (26). In general the *relative size* $[n_k/n_l]^{1/2}$ plays a key role in transmitting interactions across different sets of entities as can be seen by dividing both sides of (26) by n_l.

We have

(27) $\hat{m}_k\equiv M_k/N_k\Longrightarrow\langle\omega_i\rangle=M'(h_k+z_k^*)/M(h_k+z_k^*)$, $i\varepsilon A_k$.

Similar arguments yield replacing $M_k\equiv\sum\omega_i$ by $\sum g(\omega_i)$ for any function g,

(28) $\sum g(\omega_i)/N_k \Longrightarrow \int g(\xi)\exp[(h_k+z_k^*)\xi]dF(\xi)/M(h_k+z_k^*),\ i\varepsilon A_k.$

These formulae for computation of limiting moments can be used to extend the applications given in the text.

2. MAXIMUM ENTROPY AND OTHER RATIONALES

The probability structures put forth in Section 1 of our paper may appear arbitrary and chosen merely for convenience. There is some justification for the particular parameterization of probability structure that we chose to use. We give several arguments below. First we deal with the idea of modelling error-prone or "noise" traders. Then we show how such probabilities arise naturally from discrete choice theory.

A natural way to model the notion of "noisy beliefs" is to choose the most random probability measure subject to constraints. For example the most random probability measure on $\Omega\equiv\{-1,1\}^N$ is the uniform measure that assigns $P(\omega)=1/2^N$ to each $\omega\ \varepsilon\ \Omega$. Explanation of this idea requires a digression into the subject of maximum entropy measures.

MAXIMUM ENTROPY MEASURES

To be precise consider the following optimization problem

(29) Maximize $[-\Sigma p(\omega)\ln(p(\omega))]$, subject to,

(30) $\Sigma p(\omega)G(\omega)=G,\ \Sigma p(\omega)=1,$

where "ln(x)" denotes the natural logarithm of x, Σ is over all $\omega\ \varepsilon\ \Omega$, and G denotes a fixed level of group sentiment. Let λ_1,λ_2 be the Lagrange multipliers associated with the two constraints in (29) by order of appearance. Then it is easy to show by setting up the Lagrangian

(31) $L\equiv\Sigma-p(\omega)\ln[p(\omega)] + \lambda_1(G-\Sigma p(\omega)G(\omega)) + \lambda_2(1-\Sigma p(\omega)),$

and differentiating to obtain,

(32) $p(\omega)=\exp[\beta G(\omega)]/Z;\ Z\equiv\Sigma_{\upsilon\varepsilon\Omega}\exp[\beta G(\upsilon)],\ \beta\equiv-\lambda_1.$

Using concavity of the function $H(x)\equiv-\ln(x)x$ on $(0,\infty)$ and linearity of the two constraints in p it is straightforward, using standard nonlinear programming theory, to show that β approaches $+\infty$ $(-\infty)$ as G approaches G^* (G_*) where G^* (G_*) denote the maximum (minimum) values of G. Note that $p(\omega)$ collapses to the most uniform measure over Ω, i.e., the IID process over Ω, when $\beta=0$. Denote this measure by π and note that $\pi(\omega)=1/2^N$, for all $\omega\ \varepsilon\ \Omega$ and note that (32) may be equivalently written by multiplying the numerator by $\pi(\omega)$ and each term of the denominator by $\pi(\upsilon)$. This is useful in Ellis's (1985) development of the limit theory which we follow. Also note that Ellis's β is absorbed in our J,h. To put it another way Ellis's $\beta J,\beta h$ correspond to our J,h.

RATIONALE FOR ENTROPY MAXIMIZATION

At this point we must further digress to discuss the rationale for entropy maximization. The motivation of entropy maximization stems from my

own attempt to re-formulate the "Harsanyi" doctrine or "common priors" assumption in such a way that some diversity of beliefs is allowed at a cost of a minimal number of free parameters.

The Harsanyi doctrine is controversial. Witness the labor expended defending it by Aumann against the flat statement by Kreps: "This assumption has very substantial implications for exchange among agents; we will encounter some of these later in the book. I leave it to others to defend this assumption--see, for example Aumann (1987, section 5)--as I cannot do so. But the reader should be alerted to this modeling assumption, which plays an important role in parts of modern microeconomic theory; it is called both the *common prior assumption* and the *Harsanyi doctrine*." (1990, p. 111). Kurz (1990), for example, makes a strong argument that diversity of beliefs will remain in the face of learning in a context where one would expect belief convergence.

In view of this conflict in the profession we propose a compromise. Entropy maximization subject to constraints is given a very spirited defense as a useful way to do prediction in statistical mechanics by E. T. Jaynes (1983) and there may be a useful analogy in economics as discussed by Zellner (1991). It may possibly be viewed as a way to allow some diversity in beliefs without emptying the theory of predictive content and in Bayesian literature as a way of giving some "objectivity" to "subjective" beliefs. I use it here to motivate an anaytically tractable model of interactive group formation of beliefs or sentiment. That is to say the group is assumed to have the most random set of group beliefs subject to a given mean level G. This restriction parsimoniously parameterizes the beliefs by three parameters (β, J, h) where β is fixed by G.

If the reader does not care for the maximum entropy argument the same probabilities may be derived, as in Section 4.1, by viewing the group of interactive noise traders as solving the "social discrete stochastic choice problem"

$$(33) \quad \underset{\omega \,\varepsilon\, \Omega}{\text{Maximize}}\ G(\omega) + \mu e(\omega), \quad \beta \equiv \mu^{-1}.$$

where $\{e(\omega)\}$ is IID extreme value distributed. It is pointed out in Manski and McFadden (1980) that Prob{choose ω} is exactly equal to the logit probability (32). Since the probabilities are logit we have access to the extensive econometric literature on estimation of logit systems. Indeed this is a main part of the motivation for the type of theory we are building. More will be said about estimation in future work.

REFERENCES

(Note: Not all of the references are cited in the text. Since part of the purpose of this paper is to help give readers a guide to the literature, I have listed many references here. The Brock, Hsieh, LeBaron book cited below gives many references which are not cited here.)

Antoniewicz, R., (1991), "An Empirical Anaysis of Stock Returns and Volume," PhD Thesis, Department of Economics, The University of Wisconsin.

Anderson, P., Arrow, K., Pines, D., (1988), THE ECONOMY AS AN EVOLVING COMPLEX SYSTEM, Santa Fe Institute Studies in the Sciences of Complexity, Volume V, Addison Wesley, Redwood City, California.

Anderson, S., de Palma, A., Thisse, J., (1993), DISCRETE CHOICE THEORY OF PRODUCT DIFFERENTIATION, Cambridge: MIT Press.

Aoki, M., Miyahara, Y., (1991), "Stochastic Aggregation and Dynamic Field Effects," Deparment of Economics, UCLA, Los Angeles, CA., 90024-1477.

Arthur, B., (1992), "On Learning and Adaptation in the Economy," Food Research Institute, Stanford University.

Bak, P., Chen, K., Scheinkman, J., Woodford, M., (1992a), "Self Organized Criticality and Fluctuations in Economics," Santa Fe Institute W.P. # 92-04-018, Santa Fe, New Mexico.

Bak, P., Chen, K., Scheinkman, J., Woodford, M., (1992b), "Aggregate Fluctuations from Independent Sectoral Shocks: Self-Organized Criticality in a Model of Production and Inventory Dynamics," (revised version of 1992a).

Barnett, W., Gallant, A., Hinich, M., Jensen, M., (1992), "Robustness of Nonlinearity and Chaos Test to Measurement Error, Inference Method, and Sample Size," Department of Economics, Washington University, St. Louis, July.

Benhabib, J., ed., (1992), CYCLES AND CHAOS IN ECONOMIC EQUILIBRIUM, Princeton: Princeton University Press.

Berliner, L., (1992), "Statistics, Probability, and Chaos," with comments by Cutler, Geweke, Granger, Griffeath, Smith, Tsay, and rejoinders by Chatterjee, Yilmaz, Berliner; STATISTICAL SCIENCE, Vol. 7, #1, 69-122.

Blume, L., (1991), "The Statistical Mechanics of Strategic Interaction," Department of Economics, Cornell University, Uris Hall, Ithaca, NY 14850.

Blume, L., Easley, D., (1992), "Evolution and Market Behavior," JOURNAL OF ECONOMIC THEORY, Vol. 58, #1, October, 9-40.

Bollerslev, T., Chou, R., Kroner, K., (1992), "ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence," JOURNAL OF ECONOMETRICS, 52, 5-60.

Brock, W., (1982), "Asset Prices in A Production Economy," in J. McCall, (ed.), THE ECONOMICS OF INFORMATION AND UNCERTAINTY, Chicago: The University of Chicago Press.

Brock, W., Dechert, W., Scheinkman, J., LeBaron, B., (1990), "A Test for Independence Based Upon the Correlation Dimension," Department of Economics, The University of Wisconsin, Madison, The University of Houston, and The University of Chicago.

Brock, W., Lakonishok, J., LeBaron, B., (1990), "Simple Technical Trading Rules and the Stochastic Properties of Stock Returns," Social Systems Research Institute W.P. #9022, The University of Wisconsin, Madison; JOURNAL OF FINANCE (forthcoming).

Brock, W., (1991a), "Understanding Macroeconomic Time Series Using Complex Systems Theory," STRUCTURAL CHANGE AND ECONOMIC DYNAMICS, Vol.2, #1, 119-141.

Brock, W., (1991b), "Causality, Chaos, Explanation, and Prediction in Economics," in Casti, J., Karlqvist, A., eds., BEYOND BELIEF: RANDOMNESS, PREDICTION, AND EXPLANATION IN SCIENCE, Boca Raton, Florida: CRC Press.

Brock, W., Hsieh, D., and LeBaron, B., (1991), NONLINEAR DYNAMICS, CHAOS, AND INSTABILITY: STATISTICAL THEORY AND ECONOMIC EVIDENCE, Cambridge: MIT Press.

Brock, W., and Dechert, W., (1991), "Non-Linear Dynamical Systems: Instability and Chaos in Economics," in Hildenbrand, W., Sonnenschein, H., HANDBOOK OF MATHEMATICAL ECONOMICS, Vol. IV, Amsterdam: North Holland, 2209-2235.

Brock, W., and Potter, S., (1992), "Nonlinear Time Series and Macroeconomics," in C. R. Rao, ed. HANDBOOK OF STATISTICS, New York: North Holland (forthcoming).

Campbell, J., Grossman, S., Wang, J., (1991), "Trading Volume and Serial Correlation in Stock Returns," Princeton University, Wharton School, Sloan School, MIT, QUARTERLY JOURNAL OF ECONOMICS (forthcoming).

Casdagli, M., and Eubank, S., (1991), PROCEEDINGS OF THE 1990 NATO WORKSHOP ON NONLINEAR MODELING AND FORECASTING, Santa Fe Institute Series, Redwood City, California: Addison-Wesley.

Casti, J., (1992), REALITY RULES, Volumes I, II, New York: John Wiley and Sons.

Diebold, F., Nason, J., (1990), "Nonparametric Exchange Rate Prediction," JOURNAL OF INTERNATIONAL ECONOMICS, 28, 315-332.

de Jong, R., (1992), "The Bierens Test Under Data Dependence," Department of Econometrics, Free University, De Boelelaan 1105, 1081HV, Amsterdam, The Netherlands.

de Lima, P., (1992a), "Testing Nonlinearities Under Moment Condition Failure," Department of Economics, Johns Hopkins University.

de Lima, P., (1992b), "On the Robustness of Nonlinearity Tests to Moment Condition Failure," Department of Economics, John Hopkins University.

De Long, J., Shleifer, A., Summers, L., Waldmann, R., (1990), "Noise Trader Risk in Financial Markets," JOURNAL OF POLITICAL ECONOMY, 98, #4, 703-738.

Durlauf, S., (1989a), "Locally Interacting Systems, Coordination Failure, and

The Behavior of Aggregate Activity," Department of Economics, Stanford University.

Durlauf, S., (1989b), "Output Persistence, Economics Structure and the Choice of Stabilization Policy," BROOKINGS PAPERS ON ECONOMIC ACTIVITY, 2, 69-116.

Durlauf, S., (1991a), "Nonergodic Economic Growth," Stanford Institute for Theoretical Economics, TR # 7., REVIEW OF ECONOMIC STUDIES (forthcoming).

Durlauf, S., (1991b), "Path Dependence in Aggregate Output," Stanford Institue for Theoretical Economics, TR #8.

Durlauf, S., and Johnson, P., "Local Versus Global Convergence Across National Economies," Department of Economics, Stanford University.

Durlauf, S., (1992), "A Theory of Persistent Income Inequality," Department of Economics, Stanford University.

Ellis, R., (1985), ENTROPY, LARGE DEVIATIONS AND STATISTICAL MECHANICS, New York: Springer-Verlag.

Gennotte, G., Leland, H., (1990), "Market Liquidity, Hedging, and Crashes," AMERICAN ECONOMIC REVIEW.

Grandmont, J., (1992), "Expectations Driven Nonlinear Business Cycles," forthcoming in Proceedings of Stockholm Conference on Business Cycles organized by the Trade Union for Economic Research, Oxford: Oxford University Press.

Granger, C., and Terasvirta, T., (1992), MODELING DYNAMIC NONLINEAR RELATIONSHIPS, Oxford: Oxford University Press.

Hall, P., and Heyde, C., (1980), MARTINGALE LIMIT THEORY AND ITS APPLICATIONS, New York: Academic Press.

Hansen, L., and Singleton, K., (1982), "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models," ECONOMETRICA, Vol. 50, # 5, September, 1269-1286.

Haugen, R., Talmor, E., Torous, W., (1991), "The Effect of Volatility Changes on the Level of Stock Prices and Subsequent Expected Returns," JOURNAL OF FINANCE, Vol. XLVI, #3, July, 985-1007.

Hawley, D., Johnson, J., Raina, D., (1990), "Artificial Neural Systems: A New Tool for Financial Decision-Making," FINANCIAL ANALYSTS JOURNAL, November-December, 63-72.

Hellwig, M., (1980), "On the Aggregation of Information in Competitive Markets," JOURNAL OF ECONOMIC THEORY, Vol. 22, #3, June, 477-498.

Hellwig, M., (1982), "Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example," JOURNAL OF ECONOMIC THEORY, Vol. 26, #2, April, 279-312.

Hommes, C., (1991), CHAOTIC DYNAMICS IN ECONOMIC MODELS: SOME SIMPLE CASE-STUDIES, Groningen, The Netherlands: Wolters-Noordhoff.

Holland, J., (1992), ADAPTATION IN NATURAL AND ARTIFICAL SYSTEMS, Cambridge: MIT Press.

Hotz, V., Miller, R., Sanders, S., Smith, J., (1992), "A Simulation Estimator for Dynamic Models of Discrete Choice," NORC W.P. # 92-1, The University of Chicago.

Hsieh, D., (1987), "Testing for Nonlinear Dependence in Daily Foreign Exchange Rates," JOURNAL OF BUSINESS, Vol. 62, #3, 339-368.

Hsieh, D., (1991), "Chaos and Nonlinear Dynamics: Applications to Financial Markets," JOURNAL OF FINANCE, Vol. XLVI, #5, December, 1839-1877.

Hsieh, D., (1990), "Implications of Nonlinear Dynamics for Financial Risk Management," Fuqua School of Business, Duke University, forthcoming in JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS.

Jaynes, E., (1983), PAPERS ON PROBABILITY, STATISTICS, AND STATISTICAL PHYSICS, R., D., Rosenkrantz, ed., D. Reidel Publishing Company: Boston.

Kac, M., (1968), "Mathematical Mechanisms of Phase Transitions," In Chretien, M., Gross, E., Deser, S., (eds.) STATISTICAL PHYSICS: PHASE TRANSITIONS AND SUPERFLUIDITY, Vol. 1, pp. 241-305. Brandeis University Summer Institute in Theoretical Physics, 1966.

Kreps, D., (1990), A TEXT IN MICROECONOMIC THEORY, Princeton: Princeton University Press.

Kurz, M., (1990), "On the Structure and Diversity of Rational Beliefs," Department of Economics, Stanford University.

Kurz, M., (1991), "On Rational Belief Equilibria," Department of Economics, Stanford University.

Kurz, M., (1992), "Asset Prices with Rational Beliefs," Department of Economics, Stanford University.

LeBaron, B., (1992a), "Some Relations Between Volatility and Serial Correlation in Stock Market Returns," JOURNAL OF BUSINESS, LXV, 199-219.

LeBaron, B., (1992b), "Persistence of the Dow Jones Index on Rising Volume," Social Systems Research Institute, W.P.# 9201, The University of Wiscinsin, Madison.

Lucas, R., (1978), "Asset Prices in an Exchange Economy," ECONOMETRICA, 46, 1429-1445.

Lukacs, E., (1975), STOCHASTIC CONVERGENCE, New York: Academic Press.

Krasner, S., ed., (1990), THE UBIQUITY OF CHAOS, American Association for the Advancement of Science, Publication # 89-15S, 1333 H Street, N.W., Washington, D.C., 20005.

Kuznetsov, Y., Muratori, S., Rinaldi, S., (1992), "Bifurcations and Chaos in a Periodic Predator-Prey Model," INTERNATIONAL JOURNAL OF BIFURCATIONS AND CHAOS, Vol. 2, #1, 117-128.

Loretan, M., and Phillips, P., (1992), "Testing the Covariance Stationarity of Heavy-Tailed Time Series: An Overview of the Theory with Applications to Several Financial Datasets," SSRI Working Paper 9208, University of Wisconsin, Madison.

Majumdar, M., Mitra, T., (1992), "Robust Ergodic Chaos in Discounted Dynamic Optimization Models," Department of Economics, Cornell University.

Mandelbrot, B., (1963), "The Variation of Certain Speculative Prices," JOURNAL OF BUSINESS, 36, 394-419.

Mandelbrot, B., (1972), "Statistical Methodology for Nonperiodic Cycles: From the Covariance to R/S Analysis," ANNALS OF ECONOMICS AND SOCIAL MEASUREMENT, July, 259-90.

Manski, C., (1992), "Identification Problems in the Social Sciences," SSRI W.P. #9217, The University of Wisconsin, Madison.

Manski, C., and McFadden, D., (1981), STRUCTURAL ANALYSIS OF DISCRETE DATA WITH ECONOMETRIC APPLICATIONS, Cambridge: MIT Press.

Marimon, R., (1989), "Stochastic Turnpike Property and Stationary Equilibrium, JOURNAL OF ECONOMIC THEORY, 47: 282-306.

McFadden, D., (1981), "Econometric Models of Probabilistic Choice," Chapter 5 of Manski, C., and McFadden, D., (1981), STRUCTURAL ANALYSIS OF DISCRETE DATA WITH ECONOMETRIC APPLICATIONS, Cambridge: MIT Press.

Meese, R., Rose, A., (1991), "An Empirical Assessment of Non=Linearities in Models of Exchange Rate Determination," REVIEW OF ECONOMIC STUDIES, Vol. 58 (3), #195, May, 603-619.

Mezard, M., Parisi, G., Virasora, M., (1987), SPIN GLASS THEORY AND BEYOND, Singapore: World Scientific.

Nelson, D., (1992), "Filtering and Forecasting with Misspecified ARCH Models I: Getting the Right Variance with the Wrong Model," JOURNAL OF ECONOMETRICS, 52, 61-90.

Pearce, P., (1981), "Mean-Field Bounds on the Magnetization for Ferromagnetic Spin Models," JOURNAL OF STATISTICAL PHYSICS, Vol. 25, #2, 309-320.

Pesaran, M., Potter, S., (1991), "Equilibrium Asset Pricing Models and Predictability of Excess Returns: Theory and Evidence.

Ramsey, J., Sayers, C., Rothman, R., (1988), "The Statistical Properties of Dimension Calculations Using Small Data Sets: Some Economic Applications," INTERNATIONAL ECONOMIC REVIEW, 31, 991-1020.

Ruelle, D., (1989), CHAOTIC EVOLUTION AND STRANGE ATTRACTORS, Cambridge University Press: Cambridge.

Sargent, T., (1992), "Bounded Rationality in Macroeconomics," Hoover Institution, Stanford, CA., 94305.

Scheinkman, J., (1990), "Nonlinearities in Economic Dynamics," ECONOMIC JOURNAL CONFERENCE PAPERS, Vol. 100, # 400, 33-48.

Scheinkman, J., and LeBaron, B., (1989), "Nonlinear Dynamics and Stock Returns, JOURNAL OF BUSINESS, 62, 311-337.

Sugihara, G., and May, R., (1990), "Nonlinear Forecasting as a Way of Distinguishing Chaos from Measurement Error in Time Series," NATURE, 344, 734-741.

Sugihara, G., Grenfell, B., and May, R., (1990), "Distinguishing Error from Chaos in Ecological Time Series," Scripps Institution of Oceanography, University of California, San Diego, Department of Zoology, Cambrige University, Department of Zoology, Oxford University.

Theiler, J., Eubank, S., Longtin, A., Galdrikian, B., Farmer, J., (1992), "Testing for Nonlinearity in Time Series: The Method of Surrogate Data," PHYSICA D, Vol. 58, 77-94.

Tong, H., (1990), NON-LINEAR TIME SERIES: A DYNAMICAL SYSTEMS APPROACH, Oxford: Oxford University Press.

Tsibouris, G., (1992), PhD Thesis, Department of Economics, The University of Wisconsin, Madison.

United States National Academy of Sciences, (1987), "Report of the Research Briefing Panel on Order, Chaos, and Patterns: Aspects of Nonlinearity," National Academy Press, Washington, D.C.

Vaga, T., (1990), "The Coherent Market Hypothesis," FINANCIAL ANALYSTS JOURNAL, November/December, 36-49.

Varian, H., (1985), "Divergence of Opinion in Complete Markets: A Note," JOURNAL OF FINANCE, Vol. XL, #1, 309-317.

Varian, H., (1986), "Differences of Opinion in Financial Markets," in Courtenay Stone, ed., FINANCIAL RISK: THEORY, EVIDENCE AND IMPLICATIONS, Kluwer Academic Publishers, Boston.

Wang, J., (1989), "A Model of Intertemporal Asset Prices Under Asymmetric Information," Sloan School of Management, MIT, forthcoming REVIEW OF ECONOMIC STUDIES.

Wang, J., (1991), "A Model of Competitive Stock Trading Volume," Sloan School of Managment, MIT.

Weidlich, W., (1991), "Physics and Social Science: The Approach of Synergetics," PHYSICS REPORTS, Vol., 204, #1, May, 1-163.

West, K., (1987), "A Specification Test for Speculative Bubbles," QUARTERLY JOURNAL OF ECONOMICS, August, 553-580.

Zellner, A., (1991), "Bayesian Methods and Entropy in Economics and Econometrics," in Grandy, W., and Schick, L., (eds.) MAXIMUM ENTROPY AND BAYESIAN METHODS, Kluwer Academic Publishers, 17-31.