The long memory of the efficient market

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Using data from the London Stock Exchange we demonstrate that the signs of orders obey a long-memory process. The autocorrelation function decays roughly as $\tau^{-\alpha}$ with $\alpha \approx 0.6$, corresponding to a Hurst exponent $H \approx 0.7$. The time $\tau$ is measured in terms of the number of intervening events. This is true for market orders, limit orders, and cancellations. Although the values for $\alpha$ vary from stock to stock, in the range 0.36 – 0.77, in most cases the exponents for different stocks are quite similar, and they are always less than one. This implies that the signs of future orders are quite predictable from the signs of past orders; all else being equal, this would suggest a very strong market inefficiency. We demonstrate, however, that fluctuations in signs are compensated for by anti-correlated fluctuations in transaction size and liquidity. For example, when buy orders become more likely, buy orders tend to be smaller than sell orders and buy liquidity tends to be higher than sell liquidity. By breaking down the data by institutional codes we show that some institutions display long-range memory and others don’t.

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\textsuperscript{1} Bouchaud et al. \cite{12} independently discovered the long-memory property of order signs for stocks in the Paris exchange. We thank them for acknowledging the oral presentation of our results in May 2003.
0.6, in the range $0.36 < \alpha < 0.77$. Positive autocorrelation coefficients are seen at statistically significant levels over lags of many thousand events, spanning many days. Thus the memory of the market is remarkably long.

This immediately raises a conundrum concerning market efficiency. All other things being equal, such strong long-memory behavior would imply strong predictability, easily exploitable for substantial profits. How can this be compatible with market efficiency? We show that the answer is that other properties of the market adjust in order to compensate and keep the market efficient\footnote{For a discussion of what we mean by “efficient”, see Section V.}. In particular, the relative volume of buy and sell orders, and the relative buy and sell liquidity are skewed in opposition to the imbalances in order signs. For example, suppose the long-memory of previous order signs predicts that buy orders are more likely in the near future. All things being equal, since buy orders have a positive price response, the price should go up. But the market compensates for this: When buy orders become more common, sell market orders tend to be larger than buy market orders, and volume at the best ask tends to be larger than the volume at the best bid. As a result, even though there are more buy orders, price responses are smaller for buy orders than sell orders. This happens in just the right amount, so that price returns remain roughly white, i.e. the autocorrelation of price returns rapidly decays to zero. In order to compensate and keep the market efficient, market order volume and liquidity are also long-memory processes.

This brings up the interesting question of what actually causes these long-memory properties of markets. While not answering this question, we provide a clue about the answer by making use of the institutional codes associated with each order. Some institutions show long-memory quite clearly, while others do not.

The paper is organized as follows: In Section II we give a summary of the data set, and in Section III we provide a more technical discussion of long-memory processes and the statistical techniques we use in this paper. Then in Section IV we present the evidence that these are long-memory processes, and in particular we demonstrate that these series pass stringent tests so that we can be sure that they are long-memory with a high degree of confidence. In Section V we show how other processes compensate for the long-memory of order signs, so that price changes remain roughly efficient, and show that both order size and liquidity are also long-memory processes. In Section VI we break down the orders by institution and show that the behavior of some institutions shows long-memory quite clearly, while others do not show it at all. In Section VII we discuss the implications of the long-memory properties of order flow for delayed market impact. We conclude in Section VIII, discussing some of the broader issues and the remaining questions.

## II. DATA

In order to have a representative sample of stocks we select 20 companies continuously traded at the London Stock Exchange (LSE) in the 4-year period 1999-2002. The stocks we analyzed are Astrazeneca (AZN), Bae Systems (BA.), Baa (BAA), BHP Billiton (BLT), Boots Group (BOOT), British Sky Broadcasting Group (BSY), Diageo (DGE), Gus (GUS), Hilton Group (HG.), Lloyds Tsb Group (LLOY), Prudential (PRU), Pearson (PSON), Rio Tinto (RIO), Rentokil Initial (RTO), Reuters Group (RTR), Sainsbury (SBRY), Shell Transport & Trading Co. (SHEL), Tesco (TSCO), Vodafone Group (VOD), and WPP Group (WPP). Table I gives a summary of the number of different events for the 20 stocks.

The London Stock Exchange consists of two markets, the electronic (SETS) exchange, and the upstairs market. We study only the electronic exchange. The data set we analyze contains every action by every institution participating in this exchange. In 1999 the electronic exchange contains roughly 57% percent of the total order flow, and in 2002 roughly 62% percent. It is thus always a substantial fraction of the total order flow, and is believed to be the dominant mechanism for price formation. There are several types of orders allowed by the exchange, with names such as “fill or kill” and “execute or eliminate”. To place our analysis in more useful terms we label events in terms of their net effect on the limit order book. We label any order that results in an immediate transaction an effective market order, and any order that leaves a limit order sitting in the book an effective market order.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Tick</th>
<th>Market Orders</th>
<th>Limit Orders</th>
<th>Cancellations</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>AZN</td>
<td>652</td>
<td>2,967</td>
<td>1,454</td>
<td>4,173</td>
<td></td>
</tr>
<tr>
<td>BA</td>
<td>381</td>
<td>950</td>
<td>598</td>
<td>1,299</td>
<td></td>
</tr>
<tr>
<td>BAA</td>
<td>226</td>
<td>683</td>
<td>487</td>
<td>1,397</td>
<td></td>
</tr>
<tr>
<td>BLT</td>
<td>297</td>
<td>825</td>
<td>557</td>
<td>1,679</td>
<td></td>
</tr>
<tr>
<td>BOOT</td>
<td>246</td>
<td>711</td>
<td>501</td>
<td>1,227</td>
<td></td>
</tr>
<tr>
<td>BSY</td>
<td>404</td>
<td>1,120</td>
<td>726</td>
<td>2,250</td>
<td></td>
</tr>
<tr>
<td>DGE</td>
<td>527</td>
<td>1,329</td>
<td>854</td>
<td>2,709</td>
<td></td>
</tr>
<tr>
<td>GUS</td>
<td>244</td>
<td>734</td>
<td>518</td>
<td>1,496</td>
<td></td>
</tr>
<tr>
<td>HG</td>
<td>228</td>
<td>676</td>
<td>472</td>
<td>1,377</td>
<td></td>
</tr>
<tr>
<td>LLOY</td>
<td>723</td>
<td>1,664</td>
<td>1,020</td>
<td>3,407</td>
<td></td>
</tr>
<tr>
<td>PRU</td>
<td>448</td>
<td>1,227</td>
<td>821</td>
<td>2,496</td>
<td></td>
</tr>
<tr>
<td>PSON</td>
<td>373</td>
<td>1,063</td>
<td>734</td>
<td>2,070</td>
<td></td>
</tr>
<tr>
<td>RIO</td>
<td>381</td>
<td>1,122</td>
<td>771</td>
<td>2,174</td>
<td></td>
</tr>
<tr>
<td>RTO</td>
<td>276</td>
<td>620</td>
<td>389</td>
<td>1,285</td>
<td></td>
</tr>
<tr>
<td>RTR</td>
<td>479</td>
<td>1,250</td>
<td>820</td>
<td>2,549</td>
<td></td>
</tr>
<tr>
<td>SBRY</td>
<td>284</td>
<td>805</td>
<td>561</td>
<td>1,650</td>
<td></td>
</tr>
<tr>
<td>SHEL</td>
<td>717</td>
<td>4,137</td>
<td>3,511</td>
<td>8,648</td>
<td></td>
</tr>
<tr>
<td>TSCO</td>
<td>471</td>
<td>949</td>
<td>523</td>
<td>1,943</td>
<td></td>
</tr>
<tr>
<td>VOD</td>
<td>1,278</td>
<td>2,358</td>
<td>1,180</td>
<td>4,817</td>
<td></td>
</tr>
<tr>
<td>WPP</td>
<td>399</td>
<td>1,151</td>
<td>780</td>
<td>2,330</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>9,034</td>
<td>25,441</td>
<td>17,277</td>
<td>51,752</td>
<td></td>
</tr>
</tbody>
</table>

Table I: Summary statistics of the 20 stocks we study for the period 1999-2002. The columns give the number of events of each type, in thousands. All events are “effective” events – see the discussion in the text.
limit order. A single order may result in multiple effective orders. For example, consider a crossing limit order, i.e. a limit order whose limit price crosses the opposing best price quote. The part of the order that results in an immediate transaction is counted as an effective market order, while the remaining non-transacted part (if any) is counted as an effective limit order. Finally, we will also lump together any event that results in a queued limit order being removed without a transaction, and refer to such an event as a cancellation. Henceforth dropping the modifier “effective”, we can then classify events as one of three types: market order, limit order and cancellation.

For the set of 20 stocks described above there is a total of roughly 9 million market orders, 25 million limit orders and 17 million cancellations. Throughout this paper, unless otherwise specified, we use the number of effective events as a measure of time, which we call event time. We typically do this in terms of the number of events of a given type, e.g. if we are studying market orders we measure event time in terms of the number of market orders, and if we are studying limit orders we measure event time in terms of the number of limit orders.

Trading begins each day with an opening auction. There is a period leading up to the opening auction in which orders are placed but no transactions take place. The market is then cleared and for the remainder of the day (except for occasional exceptional periods) there is a continuous auction. We remove all data associated with the opening auction, and analyze only orders placed during the continuous auction.

An analysis of the limit order placement shows that in our dataset approximately 35% of the effective limit orders are placed behind the best price (i.e. inside the book), 33% are placed at the best price, and 32% are placed inside the spread. This is roughly true for all the stocks except for SHEL for which the percentages are 71%, 18% and 11%, respectively. Moreover for all the stocks the properties of buy and sell limit orders are approximately the same.

In our dataset cancellation occurs roughly 32% of the time at the best price and 68% of the time inside the book. This is quite consistent across stocks and between the cancellation of buy and sell limit orders. As for the case of the placement of limit orders, the only significant deviation is SHEL for which the percentages are 14% and 86%.

In the following price will indicate the mid price, i.e. \( p(t) = (a(t) + b(t))/2 \) where \( a(t) \) and \( b(t) \) are the best ask and best bid prices at time \( t \), respectively.

III. REVIEW OF METHODS FOR UNDERSTANDING LONG-MEMORY PROCESSES

A. Definitions of long-memory

There are several way of characterizing long-memory processes. A widespread definition is in terms of the autocovariance function \( \gamma(k) \). We define the process as long-memory if in the limit \( k \to \infty \)

\[
\gamma(k) \sim k^{-\alpha} L(k)
\]

(1)

where \( 0 < \alpha < 1 \) and \( L(x) \) is a slowly varying function\(^3\) at infinity. The degree of long-memory dependence is given by the exponent \( \alpha \); the smaller \( \alpha \), the longer the memory.

Long-memory is also discussed in terms of the Hurst exponent \( H \), which is simply related to \( \alpha \). For a long-memory process \( H = 1 - \alpha/2 \) or \( \alpha = 2 - 2H \). Short-memory processes have \( H = 1/2 \), and the autocorrelation function decays faster than \( k^{-1} \). A positively correlated long-memory process is characterized by a Hurst exponent in the interval \((0.5, 1)\). The use of the Hurst exponent is motivated by the relationship to diffusion properties of the integrated process. For normal diffusion, whose increments do not display long-memory, the standard deviation asymptotically increases as \( t^{1/2} \), whereas for diffusion processes with long-memory increments the standard deviation asymptotically increases as \( t^H L(t) \), with \( 1/2 < H < 1 \), and \( L(t) \) a slow-varying function.

Yet another equivalent definition of long-memory dependence can be given in terms of the behavior of the spectral density for low frequencies. A long-memory process has a spectral density which diverges for low frequencies as

\[
g(f) \simeq f^{1-2H} L(f),
\]

(2)

where \( f \) is the frequency, and \( L(f) \) is a slowly varying function in the limit \( f \to 0 \). This follows immediately from the fact that the autocorrelation and the spectral density are Fourier transforms of each other.

B. Statistical tests for long-memory

The empirical determination of the long-memory property of a time series is a difficult problem. The basic reason for this is that the strong autocorrelation

\(^3\) \( L(x) \) is a slowly varying function [13] if \( \lim_{x \to \infty} L(tx)/L(x) = 1 \). In the definition above, and for the purposes of this paper, we are considering only positively correlated long-memory processes. Negatively correlated long-memory processes also exist, but the long-memory processes we will consider in the rest of the paper are all positively correlated.
of long-memory processes makes statistical fluctuations very large. Thus tests for long-memory tend to require large quantities of data and can often give inconclusive results. Furthermore, different methods of statistical analysis often give contradictory results. In this section we review two such tests and discuss some of their properties. In particular we discuss the classical R/S test, which is known to be too weak, and Lo’s modified R/S test, which is known to be too strong.

The basic idea behind the classical R/S test [14–16] is to compare the minimum and maximum values of running sums of deviations from the sample mean, renormalized by the sample standard deviation. For long-memory processes the deviations are larger than for non-long-memory processes. The classical R/S test has been proven to be too weak, i.e., it tends to indicate a time series has long-memory when it does not. In fact, Lo [3] showed that even for a short-memory process, such as a simple AR(1) process, the classical R/S test does not reject the null hypothesis of short-memory. This fact motivated Lo to introduce a test based on a modified R/S statistic which is sensitive to true long-memory properties [3].

We now describe Lo’s modified R/S test. Consider a sample time series \( X_1, X_2, \ldots, X_n \) with sample mean \( (1/n) \sum_j X_j \) as \( \bar{X}_n \). Let \( \sigma_x^2 \) and \( \hat{\gamma}_x \) be the sample variance and autocovariance. The modified rescaled range statistic \( Q_n(q) \) is defined by

\[
Q_n(q) \equiv \frac{1}{\hat{\sigma}_n(q)} \left[ \max_{1 \leq k \leq n} k \left( \frac{1}{n} \sum_{j=1}^k (X_j - \bar{X}_n) \right) - \min_{1 \leq k \leq n} k \left( \frac{1}{n} \sum_{j=1}^k (X_j - \bar{X}_n) \right) \right]
\]

where

\[
\hat{\sigma}_x^2(q) \equiv \sigma_x^2 + 2 \sum_{j=1}^q \omega_j(q) \hat{\gamma}_j, \quad \omega_j(q) \equiv 1 - \frac{j}{q + 1} \tag{4}
\]

and \( q < n \). It is worth noting that \( Q_n(q) \) differs from the classical R/S statistics of Mandelbrot only in the denominator. In the classical R/S test \( \hat{\sigma}_n(q) \) is replaced by the sample standard deviation \( \hat{\sigma}_x \).

The optimal value of \( q \) to be used in Eq. (3) to compute \( Q_n \) must be chosen carefully. Lo suggested the value \( q = [k_n] \) where

\[
k_n \equiv \left( \frac{3n}{2} \right)^{\frac{2}{3}} \left( \frac{2\hat{\rho}}{1 - \hat{\rho}^2} \right)^{\frac{2}{3}} \tag{5}
\]

where \([k_n]\) indicates the greatest integer less than or equal to \( k_n \) and \( \hat{\rho} \) is the sample first-order autocorrelation coefficient of the data. Lo was able to prove that if the process has finite fourth moment and it has a short-memory dependence (and satisfies other supplementary conditions) \( V_n \equiv Q_n/\sqrt{n} \) tends asymptotically to a random variable distributed according to

\[
F_{V}(v) = 1 + 2 \sum_{k=1}^{\infty} (1 - 4k^2v^2)e^{-2(kv)^2}. \tag{6}
\]

This result makes it possible to find the boundaries of a given confidence interval under the null hypothesis that the time series is short-memory. When \( V_n \) is outside the interval \([0.809, 1.862]\), we can reject the null hypothesis of short range dependence with 95\% confidence.

Recently Taqqu and coworkers [17] have shown that Lo’s rescaled R/S test is too severe. In fact they showed numerically that even for a synthetic long-memory time series with a moderate value of the Hurst exponent (like \( H = 0.6 \)) the Lo test cannot reject the null hypothesis of short range dependence. For our results here we are lucky that we are able to pass the rescaled R/S test for long-memory, but the stringency of this test should be borne in mind in evaluating our results.

C. Methods of measuring the Hurst exponent

The determination of the Hurst exponent of a long-memory process is not an easy task, especially when one cannot make any parametric assumptions about the investigated time series. Several heuristic methods have been introduced to estimate the Hurst exponent. Recently some authors suggested the use of a “portfolio” of estimators instead of relying on a single estimator which could be biased by the property of the time series under investigation [18]. In this paper we will discuss four widespread Hurst exponent estimators which we describe below. These methods are the periodogram method, the R/S method, Detrended Fluctuation Analysis and the fit of the autocorrelation function. We find that the first three methods give reasonable agreement both in real and in surrogate time series. The fourth method appears to be more noisy and less reliable.

To use the periodogram method, one first calculates the periodogram \( I(f) \), which is an estimate of the spectral density.

\[
I(f) = \frac{1}{2\pi n} \left| \sum_{j=1}^{n} X_j e^{ijf} \right|^2 \tag{7}
\]

where, as before, \( n \) is the size of the sample \( X_j \). Then a regression of the logarithm of the periodogram against the logarithm of \( f \) for small values of \( f \) gives a coefficient, which is an estimate of \( 1 - 2H \) (see Eq. 2). We make our regression on the lowest 10\% of the data [18].

The second method is the R/S method [15, 16]. A description of the method, which is strongly based on R/S statistics, can be found in [1]. In summary, we divide a time series of length \( n \) in \( K \) blocks of size \( n/K \). Then for each lag \( k \) we compute the classical R/S statistics (i.e., Eq. (3) with \( \hat{\sigma}_x \) instead of \( \hat{\sigma}_n(q) \) in the denominator), starting at points \( k_i = in/K + 1, i = 1, 2, \ldots \) such that \( k_i + k < n \). When \( k < n/K \), one obtains \( K \) different values of the R/S statistics. We chose logarithmically spaced values of \( k \) and we plot the value of the R/S statistics versus \( k \) in double logarithmic scale. The parameter \( H \) is obtained by fitting a power-law relationship.
The third method is the Detrended Fluctuation Analysis [19]. The time series is first integrated. The integrated time series is divided into boxes of equal length \( m \). In each box, a least squares line is fit to the data (representing the trend in that box). The \( y \) coordinate of the straight line segments is denoted by \( y_m(k) \). Next, we detrend the integrated time series, \( y(k) \), by subtracting the local trend, \( y_m(k) \), in each box. The root-mean-square fluctuation of this integrated and detrended time series is calculated by

\[
F(m) = \sqrt{\frac{1}{n} \sum_{k=1}^{n} [y(k) - y_m(k)]^2}
\]  

This computation is repeated over all time scales (box sizes) to characterize the relationship between \( F(m) \) and the box size \( m \). Typically \( F(m) \) will increase with box size \( m \). The Hurst exponent is obtained by fitting \( F(m) \) with a relation \( F(m) \propto m^H \). The proposers of this method claim that it is able to remove local trends due to bias in the enhanced occurrence of a class of events [19].

A fourth method is to simply compute the autocorrelation function and measure \( \alpha = 2 - 2H \) by regressing the autocorrelation function with a power law. This method, however, suffers from the problem that the sample errors in adjacent autocorrelation coefficients are strongly correlated, and so this method is less accurate than the other two methods discussed above. Thus, we only use this method as an indication. Based on tests on real and surrogate data, we find that the first three methods all give very similar results; we use either the R/S method or the periodogram method when we want to get accurate values of the exponent \( \alpha \).

IV. DEMONSTRATION OF LONG-MEMORY FOR ORDER SIGNS

A. A quick look at the autocorrelation function

We consider the symbolic time series obtained in transaction time by replacing buy orders with +1 and sell orders with −1 irrespective of the volume (number of shares) in the order. This can be done for market orders, limit orders, or cancellations. As we will see, all of these series show very similar behavior. We reduce these series to ±1 rather than analyzing the signed series of order sizes \( \omega_t \) in order to avoid problems created by the large fluctuations in order size; analysis of the signed series of order sizes produces results that do not converge very well.

Figure 1 shows the sample autocorrelation functions of the order sign time series for Vodafone in the period 1999-2002 in double logarithmic scale. Vodafone was chosen to illustrate the results in this paper because it is one of the most capitalized and most heavily traded stocks in

FIG. 1: Autocorrelation function of sequences of order signs for Vodafone in the period 1999-2002 in double logarithmic scale for (a) market orders, (b) limit orders and (c) cancellations. The lag is measured in terms of the number of events of each type, e.g., number of market orders, number of limit orders, etc. In each case the autocorrelation function remains positive over periods much longer than the average number of events in a day.
the LSE during this period; however, we see very similar results for all the other stocks in our dataset. The autocorrelation function for market orders, limit orders and cancellations decays roughly linearly over more than 4 decades, although with some break in the slope for limit orders and cancellations. This suggests that a power-law relation $\rho(k) \sim k^{-\alpha}$ is a reasonable asymptotic approximation for the empirical autocorrelation function. Of course, for larger lags there are fewer independent intervals, and the statistical fluctuations are much larger.

Estimating $\alpha$ from the sample autocorrelation using ordinary least squares fit gives $\alpha = 0.39$ for market orders. For limit orders there appears to be a break in the slope, with an exponent roughly 0.4 for lags less than roughly 500 and 0.6 for larger lags. There is a similar break in the slope for cancellations, with a slope roughly 0.4 for less than 50 lags and 0.7 for larger lags. As already mentioned, the sample autocorrelation is a poor method for estimating $\alpha$, and should only be considered an indication; later on we will use more reliable estimators. But the fact that $\alpha$ is much smaller than 1 in every case suggests that these might be a long-memory processes [1]. The memory is quite persistent, as is evident from the fact that the sample autocorrelations remain positive over a very long span of time. The average daily number of market orders for Vodafone in the investigated period is approximately 1,300, whereas the slow decay of the autocorrelation function in Fig. 1 is seen for lags as large as 10,000. This indicates that the long-memory property of the market order placement is not just an intra-day phenomenon, but rather spans multiple days, persisting on a timescale of more than a week. Similar statements are true for limit orders and cancellations. See also Section IV E, where we analyze this phenomenon in real time rather than event time.

B. Statistical evidence for long-memory

In order to test the presence of long-memory properties in the time series of market order signs both longitudinally (i.e. analyzing a stock for different time periods) and cross-sectionally (i.e. analyzing different stocks) we proceed as follows. We consider the set of 20 highly capitalized stocks described in Section 2 for the 4 year period 1999-2002. Since the number of orders is different for different stocks in different calendar years, we divide the data for each year and for each stock into subsets in such a way that each set contains roughly a fixed number of orders\(^4\). To each set we apply the Lo test based on modified R/S statistics, obtaining a value for the statistics $Q_n$. Since our time series consists of +1 and −1 we do not have problems with the existence of moments. Figure 2 shows the histogram of the 324 values of $Q_n$ for the subsets of market orders. For 289 (89.2%) subsets we can reject the null hypothesis of short-memory processes with 95% confidence. Repeating this test for limit orders and cancellations gives even stronger results: For limit orders, based on 468 subsets the short-memory hypothesis is rejected at the 95% level in 97% of the cases, and for cancellations using 558 subsets it is rejected in 96% of the cases. We can therefore conclude that these order sign time series are almost certainly long-memory processes. This result is even stronger when one considers the severity of this test, as pointed out in [17].

C. Estimating the Hurst exponents

Now that we have established that these are long-memory processes we determine the Hurst exponent $H$ to see if there is consistency in the exponent in different years and for different stocks. Recall that for a long-memory process the Hurst exponent is related to

\(^4\) The number of market orders ranges from 26,438 for WPP in 1999 to 415,392 for VOD in 2002, the number of limit orders ranges from 51,798 in 1999 for BLT to 2,552,410 for SHEL in 2002, and the number of cancellations ranges from 29,395 for BLT in 1999 to 2,529,526 for SHEL in 2002. We thus divided the market orders into 324 subsets ranging in size from 25,000 to 49,999; we divided limit orders into 468 subsets ranging in size from 50,000 to 99,999, and we divided cancellations into 558 subsets in size ranging from 29,000 to 57,999.
the exponent $\alpha$ of the autocorrelation function through $\alpha = 2 - 2H$.

The first estimator we used for the determination of the Hurst exponent is least squares fitting of the periodogram. The mean estimated value of the Hurst exponent is $H = 0.695 \pm 0.039$ for market orders, $H = 0.716 \pm 0.054$ for limit orders, and $H = 0.768 \pm 0.059$ for cancellations, where the error is the standard deviation. The histograms of the exponents obtained in this way are shown in Fig. 3. We see that in every case the Hurst exponent is roughly peaked around the value $H = 0.7$ which corresponds to $\alpha = 0.6$.

Following the suggestion of [18] we also estimate the Hurst exponent for market orders through the classical R/S method [1]. In this case the mean Hurst exponent is $0.696 \pm 0.032$, which is consistent with the value obtained with the periodogram method. Figure 3(a) gives a comparison of the results of the two methods. In the inset we plot the Hurst exponent obtained from the periodogram against the Hurst exponent obtained from the R/S method, showing that the results are quite correlated on a case-by-case basis, with no discernable bias.

D. Idiosyncratic variation of the Hurst exponents

The previous results bring up the interesting question of whether there are real variations in the Hurst exponents, or whether they have a universal value, and the variations that we see are just sample fluctuations. To compare the longitudinal and cross-sectional variations we perform a classical ANOVA test. We assume that for each stock $i$ the value of the Hurst exponent in different time periods is normally distributed with mean $m_i$ and standard deviation $\sigma$. We test the null hypothesis that all the $m_i$ are equal. We indicate with $H_{ij}$ the estimated Hurst exponent of stock $i$ in sub-period $j$. There are $r = 20$ stocks, each of them with a variable number $n_i$ of sub-periods. The total number of subsets is $n = \sum_i n_i$. As usual the sum of squares of deviations of $H_{ij}$ can be decomposed in the sum of squared deviations within groups (i.e. stocks) $(n-r)s_X^2 = \sum_{i=1}^r \sum_{j=1}^{n_i} (H_{ij} - \bar{H})^2$ and the sum of squared deviations between groups $(r-1)s_1^2 = \sum_{i=1}^r (\bar{H}_i - \bar{H})^2$, where $\bar{H}$ is the sample mean for the entire sample and $\bar{H}_i$ is the sample mean for stock $i$. Under the above null hypothesis, the sum of squared deviations within groups has a $\chi^2$ distribution with $n-r$ degrees of freedom. Likewise the sum of squared deviations between groups has a $\chi^2$ distribution with $r-1$ degrees of freedom. Therefore the logarithm of the ratio between $s_1$ and $s_2$ has Fisher’s $Z$-distribution with $(r-1, n-r)$ degrees of freedom. For all the three types of orders we reject the null hypothesis with 99% confidence. Moreover in all three cases $s_1 > s_2$, showing that the cross-sectional variation of the Hurst exponent is significantly larger than the longitudinal variation, which suggests that the variations in the exponents between stocks are statistically significant. Nonetheless,

![Graphical representation of the Hurst exponent analysis](image)

**FIG. 3:** Histogram of Hurst exponents for (a) market orders, (b) limit orders, and (c) cancellations, for subsets of the data as described in the text. In (a) we show histograms for both the periodogram (continuous line) and the R/S method (dashed line), while (b) and (c) show the periodogram method only. The inset of (a) plots the results from the two methods against each other (periodogram on the $x$-axis and R/S on the $y$ axis).
it is interesting that these variations are relatively small.

E. Order flow in real time

We have shown that the sequence of order signs is a long-memory process in event time. In this section we briefly consider the correlation properties in real time. This is complicated by the fact that trading is not homogeneous. There are both strong intra-day periodicities, e.g. volume tends to increase near the open and the close, and also strong temporal autocorrelations in number of trades. Thus the number of trades in any given time interval $T$ can vary dramatically depending on the interval.

To understand the long-memory of orders in real time, we are seeking a quantity that gives information about imbalances in the signs of orders, but which is independent of the number of orders placed in a given time interval. We use two methods, which give similar results. Let $n_b(t)$ and $n_s(t)$ indicate the number of buy and sell market orders, respectively, in a time interval of length $T$ starting at time $t$. The first method follows a majority rule, which assigns the value +1 if $n_b(t) > n_s(t)$ and the value −1 if $n_b(t) < n_s(t)$. When $n_b(t) = n_s(t)$ or there are no market orders in the interval we assign the value 0. The main defect of this method is that it does not distinguish intervals with small or large imbalance of one type of orders. The second method is to use a continuous variable defined as $(n_b(t) - n_s(t))/(n_b(t) + n_s(t))$ when $n_b(t) + n_s(t) \neq 0$ and zero elsewhere. This is bounded between −1 and 1. In Figure 4 we show the autocorrelation function of $v(t)$ for a time interval $T = 5$ minutes for Vodafone. We note that a power law decay of the autocorrelation function fits the empirical data quite well, with an exponent $\alpha = 0.3$, which is close to the corresponding value in event time.

This study makes it quite clear that the long-memory properties of order signs persist across trading days. There are 102 intervals of length 5 minutes in a trading day, which means that the last lag in Figure 4 corresponds to approximately 10 trading days. Moreover the autocorrelation does not show any significant peak or break in slope near lag = 102, indicating that the long-memory properties of the market persist more or less unchanged across daily boundaries.

F. Autocorrelation of transaction volume

In this section we show that the volume of the transactions is a long memory process in event time; later on in Section VB we will argue that this is connected to the long-memory properties of order signs via market efficiency. The long-memory properties of aggregated volume have been known for a long time [8, 29]. We use modified R/S statistics in order to test the null hypothesis that the transaction volume is a short memory process in event time. The value of a stock changes in time because of the change in price. Therefore one could expect that the number of traded shares is non-stationary due to the non stationarity of the price. For this reason we decide to investigate the value of the transaction, defined as the product of the number of traded shares and the transaction price. The value is invariant under stock splits. In Figure 5 we show the autocorrelation function of the volume of Vodafone measured in terms of value in the period 1999-2002. The inset shows a histogram of the transaction volume, which is well fit by a Gamma distribution. Once we adjust for scale, this seems to be roughly the same for all the stocks in the sample (see also [30]). Moreover this result is in contrast with what has been observed for the NYSE [29] (see Section IV G).

We applied the modified R/S test to the 324 subsets used to test the long-memory properties of market order size. One of the conditions for the applicability of the modified R/S test is that the unconditional kurtosis of the time series is finite; from the inset of Figure 5 it seems that the volume distribution does not have a power-law tail, so the modified R/S test is applicable. However there could be biases in the test due to large fluctuations in volume. In 303 of the 324 subsets, or 94% of the time, we reject the null hypothesis of short-memory for the transaction volume with 95% confidence. The Hurst exponent estimated with the periodogram method varies across subsets and the mean value is $H = 0.732 \pm 0.075$, within sampling error of the exponent found for order signs.
orders. We cope with this using the Lee and Ready algorithm [32] to infer the sign of each transaction (which corresponds to the sign of the order initiating the transaction). However, this procedure is far from perfect: For NYSE data the algorithm is able to classify only about 85% of the trades. This creates problems because we do not know the sign of the remaining orders, which we classify randomly. This has some problems. First, visual inspection and correlation analysis show that the unclassified trades tend to be strongly clustered in event time. Moreover we strongly suspect that most of the trades within a given cluster have the same sign. Last, the TAQ database contains the transactions as reported by the specialist and not the individual market orders. The specialist could report a single transaction for several market orders of the same sign. These three facts substantially alter the correlation properties of the market order sign time series. Nonetheless, by random substitution of the sign of the unclassified trades, it is still quite clear that for NYSE stocks the market order sign is a long-memory process with exponents similar to those observed in the LSE.

V. MARKET INEFFICIENCY?

At first sight the long-memory property of the market order sign time series is puzzling when considered from the perspective of market efficiency. Long-memory implies strong predictability using a simple linear model. When this is combined with the fact that orders have price impact, it naively suggests that price changes should follow a long-memory process as well. That is, buy market orders tend to drive the price up, and sell market orders tend to drive it down. Thus, all other things being equal, a run of buy orders should imply future upward price movement, and a run of sell orders should imply future downward price movement. The predictability of signs is sufficiently strong that one would expect that profits could be made by taking advantage of it.

There are many ways to define market efficiency, and we should be clear how we are using it. Here we mean specifically linear efficiency, which we take to be patterns that are easily detectable by a linear time series algorithm. This allows for the possibility that there may be other more complicated nonlinear patterns, and assumes a trivial reference equilibrium of an IID random process. This is a strong efficiency in the sense of Fama [31], in that the information set is sequence of recent buy or sell order signs, which for the LSE is publicly available during this period in real time. While one might quibble about time lags in availability, since the long-memory persists for days, this is clearly not an issue.

In this section we explore the consequences of long-memory in order signs, and show that the impact of this for predictability for prices is offset by other factors. In particular, the relative size of buy and sell market orders and the relative size of the best quotes at the best bid and ask move in a way that is anti-correlated with the

G. NYSE

We performed a similar analysis on a set of stocks traded at the New York Stock Exchange. The properties of the unconditional volume distribution for individual transactions and the corresponding correlation properties were reported by Gopikrishnan et al. [29]. Their conclusion, based on the Trade and Quote database (TAQ), is that the distribution of volume of individual transactions is in the Levy regime, asymptotically decaying as \( P(V) \sim V^{-\zeta} \), where \( \zeta = 1.53 \pm 0.07 \). Moreover they found that the volume is a short-memory process. We qualitatively confirm these findings. These properties for NYSE stocks contrast strongly with those of LSE stocks. In fact, for the LSE we find the opposite set of properties: the unconditional distribution does not appear to have a power law tail for large volumes\(^5\), and the volume is a long-memory process. These differences could be due to any of several factors, for example, the different trading mechanisms in the two stock exchanges, or to the fact that we do not include upstairs trading volume in our analysis of the LSE, where as it is included (and can’t be separated) for the NYSE.

We have analyzed the correlation properties of the transaction sign time series in the NYSE. Unfortunately the TAQ database only contains quote changes and transactions, and does not give any direct information about orders. We cope with this using the Lee and Ready algorithm [32] to infer the sign of each transaction (which corresponds to the sign of the order initiating the transaction). However, this procedure is far from perfect: For NYSE data the algorithm is able to classify only about 85% of the trades. This creates problems because we do not know the sign of the remaining orders, which we classify randomly. This has some problems. First, visual inspection and correlation analysis show that the unclassified trades tend to be strongly clustered in event time. Moreover we strongly suspect that most of the trades within a given cluster have the same sign. Last, the TAQ database contains the transactions as reported by the specialist and not the individual market orders. The specialist could report a single transaction for several market orders of the same sign. These three facts substantially alter the correlation properties of the market order sign time series. Nonetheless, by random substitution of the sign of the unclassified trades, it is still quite clear that for NYSE stocks the market order sign is a long-memory process with exponents similar to those observed in the LSE.

\(^5\) Interestingly, the distribution of volumes for LSE stocks does have a power law tail at zero.
long memory of order signs, and compensates to make the market at least approximately efficient. While order signs, market order volume, and volume at the best prices are all long-memory processes, prices changes are not.

A. Inefficiency of prices in absence of liquidity fluctuations

In this subsection we show that if liquidity were fixed, the long-memory in the signs of orders would drive a strong inefficiency in prices. In the next subsection we show how market efficiency implies that liquidity must vary in order to cancel out the long-memory of order signs, which implies that liquidity is also a long-memory process.

We first construct a series of surrogate prices assuming that liquidity depends on volume but is otherwise fixed. The relation between the volume of a market order and the consequent price shift is described by the price impact function (also called the market impact function). Recent studies of the impact of a single transaction [20–23] have shown that the average market impact is a concave function of order or transaction volume, matching other studies based on time-aggregated volume [24–26]. It appears that the average impact varies across markets and stocks. For example, for a set of 1000 stocks traded at NYSE (which works with a specialist) the impact is roughly

$$E(r|V) = \frac{\text{sign}(V)|V|^\beta}{\lambda}$$  \hspace{1cm} (9)

where $r$ is the logarithmic price return, $V$ is the volume of a transaction, $\lambda$ is a liquidity parameter, and $E(.)$ indicates the expected value. The exponent $\beta$ depends on $V$ and it is approximately 0.5 for small volumes and 0.2 for large volumes [22]. The liquidity parameter $\lambda$ varies for each stock, and in general may also vary in time. Bouchaud and Potters [23] analyzed a much smaller set of stocks traded at the Paris Bourse and NASDAQ and suggested a logarithmic price impact function. For the LSE, Figure 6 shows the price impact of buy market orders for 5 highly capitalized stocks, i.e. AZN, DGE, LLOY, SHEL, and VOD. The price impact is well fit by the relation $E(r|V) \propto V^\beta$, where $V$ should now be interpreted as the market order size $V = |\omega|$ and $\beta \approx 0.3$.

If one assumes that the price impact is a deterministic function of order size, since market order placement constitutes a long-memory process, the generated price return time series must be long-memory too. We test this conclusion by constructing a synthetic price time series using real market order flow with a deterministic impact function of the form of Equation (9), but with $V$ now representing market order size. We use $\beta = 0.3$ as measured for Vodafone in Figure 6, and arbitrarily set $\lambda = 1$. For each real market order of volume $V_i$ and sign $\epsilon_i = \pm 1$ we construct the surrogate price shift $\Delta p_i = \epsilon_i V_i^{0.3}$, and construct a surrogate price series iteratively using $p_{i+1} = p_i + \Delta p_i$. In order to qualitatively test the correlation properties and the linear efficiency of this time series we use a diffusion plot. We plot the quantity $E((p_{i+\tau} - p_i)^2)$ as a function of $\tau$. Providing the series has a finite second moment, as this series does, there are three possible cases, depending on the correlation properties of the price shift\(^6\). (1) When the price shift is uncorrelated $E((p_{i+\tau} - p_i)^2)$ depends linearly on $\tau$. (2) When the price shift is a short-memory process (for example the autocorrelation decays exponentially) $E((p_{i+\tau} - p_i)^2)$ is nonlinear for $\tau$ smaller than the characteristic time scale and becomes linear for larger lags. (3) When the price shift is a long-memory process $E((p_{i+\tau} - p_i)^2) \sim \tau^{2H}$, where $H$ is the Hurst exponent. The price time series is linearly efficient when $E((p_{i+\tau} - p_i)^2)$ depends linearly on $\tau$, i.e. case (1). Figure 7 shows $E((p_{i+\tau} - p_i)^2)$ as a function of $\tau$, where $p_i$ is the synthetic price generated as described above and $t$ and $\tau$ describe time as measured by the number of events. There is clearly upward curvature, with a slope greater than one, implying that synthetic price returns are described by a long-memory process, in contradiction with the assumption of linear efficiency.

\(^6\) A random process whose second moment does not exist can have anomalous diffusion with $H \neq 0.5$ even though the autocorrelation function does not exist.
There are two possible explanations for how the real price series can be efficient when the surrogate price series defined above are inefficient. We have only used market orders above, so the first possible reason is that the price shift generated by limit orders and cancellations act to make the market efficient. The second possibility is that the assumption of a deterministic price impact is wrong and efficiency comes about due to fluctuations in the impact (and therefore in the liquidity). The first reason has been recently suggested in Ref. [12]. Their argument is that the price shift due to a market order is anticorrelated with the price shift generated by limit orders and cancellations placed between market orders. We verified empirically that such an anticorrelation does exist. However, as we will show below, the market is efficient even when we include only price shifts driven by market orders. Instead, we show that efficiency is due to fluctuations in liquidity.

To show that efficiency does not depend on limit orders and cancellations, we take the real price shift \( \Delta p_i \) due to each market order \( i \) and we iteratively construct a surrogate time series \( p_{i+1} = p_i + \Delta p_i \) starting with an arbitrary value \( p_0 \). Figure 7 shows the diffusion plot for this surrogate price time series. It is very close to linear, indicating the price series is linearly efficient. This shows that market order fluctuations are efficient even when considered by themselves, so the explanation of efficiency must lie elsewhere, as discussed in the next subsection.

**B. The key role of liquidity fluctuations in market efficiency**

Because liquidity varies strongly in time, for understanding market efficiency it is not a good approximation to describe it by a constant average value, as was done in Equation 9. In fact, studies of real data make it quite clear that the fluctuations in liquidity are large in comparison to the volume dependence of \( E[\lambda_i | V_i] \), and that in Equation 9 one should regard \( \lambda \) as a random variable whose fluctuations are at least as large in relative terms as those of \( \Delta p \) [27]. In this section we show that fluctuations in liquidity are critical for market efficiency, and that in fact fluctuations in liquidity are anti-correlated with fluctuations in order signs in just such a way as to keep the market approximately linearly efficient.

We first show that uncorrelated fluctuations in liquidity are not sufficient to ensure linear efficiency. Consider Equation 9 and let us assume that the inverse of the liquidity \( \ell_i \equiv 1/\lambda_i \) is a random variable uncorrelated with market order sign and size. In the previous section we have seen that \( a_i \equiv \epsilon_i V_i^\beta \) is a long-memory uncorrelated process. Therefore if \( E(a_i) = 0 \) then \( E(\Delta p_i) = 0 \), and the auto-covariance of price return is

\[
\gamma_{\Delta p}(\tau) = E(\Delta p_{i+\tau} \Delta p_i) = E(a_{i+\tau} a_i \ell_{i+\tau} \ell_i) = \gamma_a(\tau) (\gamma_\ell(\tau) + E(\ell^2))
\]

Now \( \ell \) is by definition a positive quantity and \( E(\ell) > 0 \). Therefore the term in brackets in the last line of Eq. (10) cannot be zero and \( \gamma_{\Delta p}(\tau) \neq 0 \), i.e. when the liquidity is uncorrelated with the order flow, the market cannot be efficient.

Now we show that fluctuations in liquidity are not random, but rather are synchronized with the long-memory of order signs in order to make the market roughly linearly efficient. To do this we need a measure of liquidity. We take as our proxy the volume at the best price: for a buy market order we take the volume at the best ask, and for a sell market order we take the volume at the best bid. This can be justified on the grounds that only very few orders penetrate more than one occupied price level [27]. We now repeat the previous experiment, constructing surrogate price series using the real order flow. We know from the previous analysis that the market order sign is a long-memory process. In panel (a) of Figure 8 we plot the autocorrelation of the sign for Vodafone (top curve). We have also seen that the correlation between sign and volume decreases the autocorrelation of the synthetic price shift \( \Delta p \propto \epsilon V^\beta \), but it is unable to remove the long-memory properties (see the diffusion plot in Figure 7). The bottom curve of Figure 8 (a) is the autocorrelation function of this synthetic time series.

When we introduce fluctuations in liquidity through the proxy of volume at the best price, the price time series becomes linearly efficient. To show this we compare the autocorrelations of three surrogate time series.

**FIG. 7:** Diffusion plot of two surrogate time series of prices. The dashed line is the diffusion plot of the surrogate time series obtained by using the real order flow of market order of Vodafone (volume and sign) and by using a deterministic price impact of Eq. (9) with \( \beta = 0.3 \). The continuous line is the diffusion plot of the surrogate time series obtained by the real price shift due only to market orders for Vodafone. The dotted line is a linear function to be used as a guide for the eye.
FIG. 8: The autocorrelation function of surrogate price shift series $\Delta p_i$ with and without varying liquidity, based on data from Vodafone. (a) shows two autocorrelation functions on double logarithmic scale, one for $\Delta p_i = \epsilon_i$ (top curve) and one for $\Delta p_i = \epsilon_i V_i^{0.3}$ (bottom curve). $\epsilon_i$ is the sign of the $i^{th}$ market order and $V_i$ is its size. In (b) the autocorrelation is plotted on linear scale, and the increments $\Delta p_i = \epsilon_i V_i^{0.3}/\lambda_i$, where $\lambda_i$ is the volume at the best price, i.e. the best ask when $i$ is a buy order and the best bid when it is a sell order. The series in (a) gives a series with long-memory, whereas in (b) the autocorrelation function becomes negative within ten lags or so. This demonstrates how liquidity, volume, and order signs interact to maintain linear market efficiency.

shown in Figure 8. The first has increments $\Delta p_i = \epsilon_i$, and clearly has long-memory. The second has increments $\Delta p_i = \epsilon_i V_i^{0.3}$, and still has long-memory, though less so than before, indicating that fluctuations in market order volume are somewhat anti-correlated with the long-memory in signs. The third has increments $\Delta p_i = \epsilon_i V_i^{0.3}/\lambda_i$, where $\lambda_i$ is the volume at the best price. The inclusion of the time varying liquidity term removes the long-memory, making the autocorrelation decrease sufficiently fast that it begins to have negative values in less than ten lags.

In more explicit terms, what does this imply about the market? For example, consider a period in which there has been a run of buy market orders. This implies that the next order is more likely to be a buy order. This is compensated for by two effects: First, the expected volume of the next buy market order is smaller than that of the next sell order. Second, it implies that the volume at the best ask is likely to be lower than the volume at the best bid, so that a buy market order of a given size will tend to generate a smaller price response than a sell market order. These two effects work together to nearly cancel the long-memory of the sign process to keep the market linearly efficient, with price increments that do not have long-memory. In describing things this way, we do not mean to necessarily suggest that the long-memory of order signs is primary, and that of volume and liquidity are consequences, but just that they are intimately related: One could equally well say that the long-memory of order signs adjusts in order to offset that of volume and liquidity. The key point is that in order to ensure linear efficiency the memory of these three processes are intimately linked. Thus, since order signs have long-memory, it is no surprise that volume and liquidity also have long-memory, as seen in Figure 9.

FIG. 9: The autocorrelation of the volume at the best prices, shown in double logarithmic scale, as a function of time measured in terms of the number of market orders. The three curves shown, from top-to-bottom, are the volume at the best ask, best bid, and best price (i.e. the best ask when the order is a buy order and the best bid when the order is a buy order). All three are long-memory processes.

VI. INDIVIDUAL INSTITUTIONS

In this section we consider the behavior of individual institutions in order to gain some understanding of what drives the long-memory processes described above. The LSE database allows us to track the actions of individual institutions through a numerical code which identifies the institution. For privacy reasons the code is different for different stocks and it is reshuffled each calendar month. Therefore our analysis will be limited to a single trading month.

We consider as a case study the market order placement of Vodafone in July 2002. Our choice is motivated by the fact that Vodafone is one of the most heavily traded stocks. In this month there were 45,774 market orders distributed across 155 trading institutions. We have found that the 12 most active institutions are responsible for more than 70% of market orders. Thus, the participation in trading is extremely inhomogeneous among the institutions [28], with few institutions placing many orders and many institutions placing only a few
orders. Table II shows the identification code, the number of market orders, and the fraction of market orders that are buy orders for each of the twelve largest institutions. In Figure 10 we show the autocorrelation function of the time series of market order signs for four active institutions. In panel (a) we show two institutions whose market order flow is a long-memory process. One of the two institutions (code 3589) is the most active institution, which placed buy market orders 24% of the time, and the other one (code 1886) placed buy market order 51% of the time. We see that in both cases the autocorrelation function is well-described by a power law with an exponent $\alpha \simeq 0.5$, which corresponds to $H = 0.75$. Panel (b) shows two active institutions (code 3007 and code 823) whose market order sign time series is a short-memory process. To test the hypothesis that the individual market order placement is a long-memory process more rigorously, we apply the modified R/S test to the time series of the market order signs of the twelve most active institutions. Table II reports the value of $Q_n$. A boldface font indicates the cases when $Q_n$ is outside the 95% confidence interval of the null hypothesis of short-memory. We see that for 7 of the 12 active institutions we reject the null hypothesis of short-memory process.

This result shows that even at the institution level the placement of orders has long-memory properties. This is not true for all institutions, but rather there is an heterogeneity in their behavior. A correlated sign in the order placement could be an indication of splitting a large order in smaller size orders in order to maximize profit without paying too much in terms of price impact. On the other hand an uncorrelated (or at least short range correlated) sign in the order placement could indicate different strategies such as, for example, market making. In section VIII A we suggest and discuss possible causes of the long-memory of order flow.

VII. IMPLICATIONS FOR MARKET IMPACT

In this section we discuss a practical consequence of long-memory of order flow. We have seen in Section V.A that the market impact is a concave function of volume. We may therefore ask about the price shift in the future (in transaction time) given that an order of a given volume and sign in the present. To be more specific, let us consider a buy market order of volume $V_1$ occurring now. The generated price shift $\Delta p_1$ is the difference between the midprice just after the order and just before the order. Between this market order and the next market order the midprice can change because of new limit orders and cancellations, generating a price change $\Delta p_2$. When the next market order arrives a new midprice shift $\Delta p_3$ occurs. The total price shift between the instant
FIG. 11: (a) A decomposition of the average delayed market impact of two successive market orders, as defined by Equation (11), for Vodafone. All four elements are conditioned on the size $V_1$ of the first market order. The immediate impact of the first market order, $E(\Delta p_1 | V_1)$, is the continuous line; the impact of any intervening limit orders or cancellations, $E(\Delta p_2 | V_1)$, has triangles. The immediate impact due to the second market order, $E(\Delta p_3 | V_1)$, is shown with squares, and the total market impact $E(\Delta p_{1-2} | V_1)$ is shown with circles. (b) The average market impact for a series of market orders, conditioned on the volume $V_1$ of the first order, $E(\Delta p_{1-m} | V_1)$. $m = 1$ (black), $m = 2$ (red), $m = 3$ (green), $m = 4$ (blue), $m = 5$ (orange) $m = 6$ (cyan) and $m = 10$ (magenta). The average market impact builds steadily with each order; this is caused by the long-memory of the order sign and order size. Both panels show the results for buy market orders just before the first order is placed and the instant just after the second market order is placed is therefore

$$\Delta p_{1-2} = \Delta p_1 + \Delta p_2 + \Delta p_3$$

(11)

If the order flow were random we would expect that $E(\Delta p_{1-2} | V_1) = E(\Delta p_1 | V_1)$ since the volume and the sign of the next orders is uncorrelated with the corresponding quantities of the first order. In Figure 11 we present a decomposition of the impact of two successive market orders in the terms described above. In panel (a) of Figure 11 we show the four quantities $E(\Delta p_1 | V_1)$ (the same quantity shown in Fig. 6), $E(\Delta p_2 | V_1)$, $E(\Delta p_{1-2} | V_1)$ and $E(\Delta p_{1-3} | V_1)$. We see that $E(\Delta p_1 | V_1)$ is almost zero, meaning that the price shift due to limit orders and cancellations after a market order is relatively unimportant. This result suggests that the role of price reversion due to limit orders and cancellations between two market orders is marginal in making the market efficient. On the other hand $E(\Delta p_2 | V_1)$ is clearly positive and increasing with $V_1$. This is due to the strong temporal correlation in market order sign and size. In fact if the first market order is a buy market order it is probable that the next market order is also a buy and the volume of the second market order is correlated with the first one. Therefore it is more probable that the price will move up due to the arrival of the second order. Figure 11 shows $E(\Delta p_{1-2} | V_1)$, which is simply the sum of the three terms, as shown in Eq. (11). The distance between $E(\Delta p_{1-2} | V_1)$ and $E(\Delta p_1 | V_1)$ is a measure of the effect of the correlation of order sign and size in the delayed price impact.

To extend this analysis to more orders, we study the delay market impact $E(\Delta p_{1-m} | V_1)$ where $m$ is the number of future market orders. Panel (b) of Figure 11 shows this quantity as a function of $V_1$ for $m = 2, 3, 4, 5, 6$ and $m = 10$. We see that for a fixed value of $V_1$, $E(\Delta p_{1-m} | V_1)$ is an increasing function of $m$. This is clearly due to the long correlation of market order sign and size. Eventually, for large values of $m$, $E(\Delta p_{1-m} | V_1)$ becomes independent of $m$.

Our results lead to conclusions that are substantially different from those of Bouchaud et al.[12]. They have suggested that placement of limit orders by market markers may offset the impact of market orders. We suggest instead that this is due to fluctuations in liquidity, and demonstrate that limit order placement plays a fairly minor role in delayed impact, at least over a short time range. We also show that the fluctuations in liquidity proxy eliminate the long memory in prices, and hence in impact, over a fairly short time range.

VIII. CONCLUSIONS

A. Possible causes of long-memory order flow

We have shown that the sign of order flow is a long-memory process. What might cause this? In this section we make a few speculations about the possible origin of long-memory.

One possible explanation for long-memory in order flow is that it simply reflects news arrival. Good (or bad) news may be clustered in time, driving the sign of order flow. Such news could either be external to the market, or it could be generated by factors internal to the market. If external it could be a property of the natural world, a reflection of the environment that humans nec-
essarily interact with. We know that floods, hurricanes, earthquakes, and natural disasters have a power law distribution, and perhaps these are just symptoms of a ubiquitous property of the natural world that is reflected in what we consider “news”. Alternatively, this could be an internal property, due to human social dynamics. Such “news” might be internally generated, e.g. due to herding behavior [33], or it might be caused by inattention: Time lags in the response of investors to news arrival can cause autocorrelations in order flow. However, it is not clear why this should have a power law distribution.

A different explanation is in terms of the execution of large orders, which leads to order splitting. It is well-known that institutions with large orders frequently split them into small pieces [34, 35], spreading out the execution of the orders over periods that can be many months long. If such orders have a power law distribution, and the time needed to fully execute an order is proportional to the size of the order, then this might give rise to power law autocorrelations in time. The idea that order size would have a power law distribution is not implausible given that many related quantities, such as firm size, wealth and mutual fund size, have power law distributions [36–38].

We have tended to discuss the autocorrelation of order signs as though this were a primary property, and the behavior of volume and liquidity are consequences, which must exist in order to maintain market efficiency. An alternative is that this reasoning is reversed, and that the autocorrelation of volume and liquidity are primary properties, and that of order signs is a consequence. However, it still remains to be determined why these should have such strong temporal autocorrelations. At this stage we simply don’t know what causes this phenomenon, but it is clearly a remarkable aspect of human economic activity that deserves more attention.

**B. Summary**

In this paper we have conclusively demonstrated that the signs of orders of 20 large British stocks are long-memory processes. This is separately true for market orders, limit orders, and cancellations. This long-memory behavior is surprising because it indicates that the signs of future orders are highly predictable. This predictability does not apply to price movements, however, because liquidity and market order size compensate in an anti-correlated manner. As a result, both volume of orders and volume at the best prices are also long-memory processes. Thus, despite the striking predictability of every feature of the market except prices, the market nonetheless appears to remain roughly linearly efficient.

Our analysis argues that the consequences of long-memory in order signs is somewhat different from that hypothesized by Bouchaud et al. [12]. The key difference concerns the way in which liquidity is treated. They assume a deterministic propagator for market impact, which amounts to assuming that liquidity is fixed. In contrast, we demonstrate that liquidity is time-varying in a manner that is anti-correlated with order signs. This, together with time variations in market order volume, are crucial in preventing delayed market impact (e.g. as shown in Figure 11) from being infinitely persistent.

We have not provided an explanation of why these long-memory properties exist. However, we do perform a breakdown of the order flow for the most active institutions, and show that some of them display the long-memory property quite clearly, while others do not show it at all. This shows that the cause is due to the behavior of certain institutions, and not others. It might, for example, be caused by strategic order splitting, to minimize transaction costs. However, this is just speculation, and in any case does not make it clear why this would cause power law autocorrelations, rather than exponential correlations with a long time constant. We intend to make a more thorough study of the institutional properties in the future [28].

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