The thermodynamic approach to the market equilibrium

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§1. The metaphors of equilibrium

The theory of economics is an example of a non-trivial case in which ontological assumptions that underlie mathematical models are mathematical models themselves or, to say more precisely, are mathematical metaphors. A basic mathematical metaphor for the classical model of the market is that of a *mechanical equilibrium*.

The basic idea of such a model is that small deviations of the system from the point of equilibrium produce "forces" which try to return the system to the equilibrium state\(^1\).

In some, very important, sense "the invisible hand" of the market in this model is equivalent to a mechanical force. The economics is considered as a dynamical system. *Time* stands as a key notion, and the mathematical structure of the economical models is represented by a system of differential equations.

In the mechanical models, the equilibrium is considered as a state at which the forces applied to the system counterbalance each other and the potential energy attains its extremum\(^1\). Consequently, to apply the mechanical metaphor of equilibrium in the economics, some analogs of the mechanical notions are needed. Such a conceptualization is not “harmless” at all: it implies that the system, having slightly digressed from the state of equilibrium, will return to this very state being left alone. As far as the dynamics of the market is concerned, the neo-classical economics inherited the classical approach. Here lie the roots of ideas how to revitalize the economics via financial stabilization, the essence of monetarian approach to vitalization of economics. According to it, it suffices to release the prices while preserving the volume of money for the system to immediately come to an equilibrium.

The practice of "shock therapies" illustrate that this is not always the case. Still, practice may lead us astray. To unearth the reasons why the economical systems refuse to come to an equilibrium as predicted, we have to deeply analyze, first at all, the "mathematical metaphor" used. The question is

are the dynamical systems adequate and sufficient a metaphor to describe the equilibrium of economical systems?

In physics, it is well-known, there are other, distinct, approaches to the conceptualization of the intuitive notion of equilibrium. Our construction is based on the thermodynamic notion of equilibrium. According to this concept, the system gets in the state of equilibrium not because it is being affected by "forces", but simply because this is the most probable state of the system, consisting of numerous parts, each of which is characterized by its independent dynamics.

This approach may refer as well to mechanical systems obeying the laws of mechanics. But, if the system is very complex, its general behavior is determined by absolutely different principles, very unlike those of mechanics.

This distinction in the mathematical description of how the system changes its state is fundamental. In terms of thermodynamic approach to equilibrium, the system, instead of evolving in time, simply changes it position in the space of macroscopic parameters, remaining on certain surface, the *surface of state*, singled out by the “equation of state”.

Time is not included into a set of parameters important for the description of the system's equilibrium. The equation of state is given by (linear) dependencies between the differentials of

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\(^1\) Clearly, we have in mind the stable equilibrium. Though a nonstable one is no less interesting, we will only consider the stable one.
macroscopic parameters, in other words, by a system of Pfaffian equation\(^2\). By changing one or several macro-parameters of the system we simply moves the system along the surface of state.

In essence, from the mathematical point of view, the investigation of equilibrium in such a model is a problem of differential topology of the surface, described by the equation of state\(^ii\).

Such a metaphor of equilibrium essentially differs from the mechanical one. Time does not occurs here as an internal parameter of the system, the parameter that determines its dynamics, but as an external one.

Dependencies between the differentials of changeable macro-parameters are determined by the internal structure of the system, for example, by the energy values of subsystems.

In the beginning of the 20-th century, C. Caratheodory proved\(^iii\) that it is possible to logically develop the thermodynamic theory, drawing, exclusively, on the assumption that "the equation of state", i.e., the surface in space of thermodynamic variables, corresponding to a system of Pfaffian equations, exists. In this case, the second law of thermodynamics is formulated as existence, in an infinitesimal neighborhood of each state of the system, of such states that cannot be reached without the change of entropy.

In other words, a simple assumption that the differentials of "generalized position" are constrained (and this constraint is nonintegrable, in Hertz's words, nonholonomic), appears to be sufficient to develop the system of thermodynamic equations. Accordingly, the idea of equilibrium will look completely differently. This thermodynamic equilibrium, unlike that from a mechanical metaphor, would mean not the existence of a singular point of a solution of a system of differential equations or an extremum of a potential function, but movement along the surface of state.

In what follows I will show that there are most serious grounds to believe that the Pfaffian equations (and, consequently, thermodynamic metaphor of equilibrium) are often more adequate for the description of various economical phenomena than the mechanical metaphor.

Under some additional assumptions on the thermodynamic system described by the Pfaffian equations the Le Chatelieu principle\(^iv\) is applicable. Namely, the system demonstrates a behavior obstructing influences exercised on it in the result of changes of the external macro-parameters.

The systems in economics also demonstrate such a behavior under certain conditions. For example, the increase of prices can be incentive for the production in order to support the consumption level; attempts to impose total control over levels of production or consumption trigger the process of corruption of executive bodies, which diminish the effect of such control, etc.

The important discovery of the past was that the phenomena of such homeostasis in physics, sometimes appearing as almost reasonable behavior, can be explained on the basis of the thermodynamic metaphor, proceeding from the very simple assumption, namely, that for the most time the system is staying in the most probable state.

In case of economics, it may look as if the system is directed, citing A. Smith, by an "invisible hand". At the time when A. Smith was working on his book, the principles of thermodynamics were not known yet, so the "invisible hand" was interpreted in terms at hand, those of a mechanical metaphor. A. Smith’s conceptual model that identifies human interests with "forces" of the market gives really serious grounds for such an interpretation.

Let us consider an “imaginary experiment” of A. Smith a bit closer. Smith assumes\(^v\) that an increase of commodity prices brings about an increase in at least one of the price-making components _ the rent, workers’ salaries or profit _ giving a signal to actors, or rather forcing them,

\(^2\) A Pfaffian equation is an equation of the type \(\omega=0\), where \(\omega\) is a differential 1-form. For a modern exposition of the theory of Pfaffian equations see.
to change their behavior (to exploit more facilities, increase the offer of jobs or to expand manufacturing), and this change of behavior finally leads to the reduction of price.

There is, however, one very weak spot in the “imaginary experiment” of A. Smith. The only immediately accessible information for the market actors is the price. Whatever changes it up or down the reasons for this change are not immediately revealed to the observer. It is most often not clear to the buyer at all if the changes in the price were due to change-making factors (say, raise or fall of profit) or the reason was in increase of demand? Usually, such information is the seller's most guarded secret.

In the studies of researchers from the Austrian school we find serious arguments in favor of the hypothesis that no complete information on the hidden components of the price is available for the market actors. This, consequently, means that they simply cannot behave in the way, described by the A. Smith’s “imaginary experiment”. In blunt terms, this “imaginary experiment” was based on false assumptions. This necessitates to put under doubt the mathematical metaphor underlying the classical concepts of the market dynamics, i.e., the mechanical metaphor.

In other words, the ontology of the classical model of economics is, indeed, not indisputable.

F. Hayek 31 despite of his vigorous diatribes of socialist ideas of regulation of economics and adherence to market principles, negated liberal viewpoints on the role of selfishness in market economy. Hayek directly asserts that the market economy is based on the observance of moral principles, ensuring survival of the community in competition with the other communities, and that these principles not infrequently directly contradict mercantile egoistic interests.

M. Weber, in his famous work about the role of protestant ethics in genesis of capitalist economy, also developed similar views. Weber shoed that the protestant ethics (the scrupulous honesty and intentions to work hard in the frames of the capitalism formed in the Northern Europe) are necessary conditions for richness growth. The interests of the members of society are being fulfilled as a result abiding the moral laws in the society as a whole. This directly contradicts the idea of A. Smith that the common well-being is a resulting corollary of individuals attempting to fulfill their egoistic interests.

In neo-classical models of the economical equilibrium constructed later, such, for example, as the ones due to Arrow-Debreu-McKenzie, in order to prove the existence of equilibrium, one assumes that the subjects of economic activity maximize their utility functions. Thus, the arguments on in-built incompleteness of the information available for the market actors were ignored.

This postulate about unavoidable incompleteness of the information seems to be one of the basic reasons why the economists of the Austrian school rejected applicability of mathematical methods in economics. It looks as though these scientists were not so much dissatisfied with the mathematics itself, but rather with the mechanical metaphor used to construct models of equilibrium the metaphor in which, in the end, the utility played the same role as the potential in the classical dynamical systems.

The dissatisfaction with such models of equilibrium was, however, discernible not only on the part of the opponents of mathematical economics, but also among its champions. The latter were anxious with the absence, within the framework of the mechanical metaphor, of satisfactory stability theory of the market equilibrium. Most profound disappointments were connected with impossibility to consider, in all cases, the excess demand as continuous function of the price. Controversial examples are well known, see, for example, the work by B. Arthur32.

One can, of course, try to “improve” the theory, remaining in the confines of the mechanical metaphor and liberal dogma. We believe, however, that although the arguments of the Austrian
School of economics are insufficient to completely reject possible applicability of mathematics in economics, they suffice to be the reason to change the mathematical metaphor of the equilibrium.

F. Hayek’s concept of market, as the process of discovery, emphasizes the key role of *information* in the market economy (as opposed to the priority of unobservable utility functions in neo-classical models, based on the mechanical metaphor of an equilibrium). In what follows we develop a model of market equilibrium proceeding from the information theory: Brillouin showed\(^{viii}\) that the mathematical information theory is, in essence, identical to thermodynamics, if we identify *information* with negative *entropy*.

The quantity of information, received at the instance of interaction of the subject with the system, is measured in the information theory by the logarithm of the *relative reduction of opportunities for choice enjoyed by the subject before the information had been obtained*. It is easy to understand that such an interpretation of the information theory directly links it to the behavioral description of the market actors, thus making entropy the most important parameter to the market interactions.

This idea is not new\(^v\). Nevertheless, until now, the idea to apply the notion of entropy to the study of economics was only realized for particular purposes (mainly, in relation to the transportation problems, and was not used with the aim to build up the whole theory of economical equilibrium).

The entropy alone is not sufficient to create the thermodynamic theory of economic equilibrium. To this end, we need the whole spectrum of thermodynamic variables, such as temperature, pressure, chemical potential, free energy, etc.

The basic idea of thermodynamic approach to the analysis of economical equilibrium is as follows. If the system is described on two ontological levels — a "macroscopic" and a "microscopic" ones — and one macroscopic state is characterized by a multitude of microscopic states (their number is called the *statistical weight* of the macroscopic state), and the system will, generally, remain in the most probable state, i.e., in the state with the greatest statistical weight.

The conditions of applicability for the thermodynamic approach can be formulated in very general terms. It becomes clear thereby, that the applicability of thermodynamics goes beyond the realm of physical systems.

One can imagine the Large System that can be decomposed, or rather consists of a huge number of Small Systems, each with its independent dynamics. We assume that the state of the Large System, and its relatively large parts, is described by a certain number of macro-parameters, which are additive, i.e., the total values of the macro-parameters appear to be the sums of values of the same macro-parameters of parts.

In physics, *energy* is an example of such a parameter if we neglect superficial interaction between the parts of the Large System. This is a basis for application of thermodynamic methods in physics. Let further each of the systems considered be additionally characterized by a set of micro-parameters which can have distinct values at the same value of the fixed macro-parameter. Their values are determined by the system's dynamics and, generally, can be of interest in relation with our task in one aspect only. Namely, having fixed them, an exact state of the system becomes known and one can answer, how many various micro-states correspond to one macro-state of the system.

At this stage, we can introduce the notion of statistical weight, as a number of micro-states corresponding to one macro-state, and the notion of entropy, as the measure of uncertainty of the macro-state of the system, which is a function of the number of micro-states.
If the system is such that the micro-states of the parts of the Large System are statistically independent, it is possible to compute the statistical weight of the Large System as a whole, given the statistical weights of all Small systems. For that purpose, it suffices to multiply the statistical weights of Small Parts.

If we wish the entropy be additive, we may regard it as the logarithm of the statistical weight. Since \( \ln(ab) = \ln a + \ln b \), it follows that the uncertainty, or entropy, is additive.

Assuming that nothing is known about the dynamics of the system, except that it is very complex, the natural assumption for the probability value of any macro-parameter is that it is proportional to the number of the appropriate micro-states, i.e., to the statistical weight. Since the logarithm is a monotonous function, the most probable state, i.e., the state with the greatest statistical weight, is, at the same time, the state with the greatest entropy, i.e., with the greatest extent of uncertainty.

This is the core of the second law of thermodynamics _the entropy tends to increase_ as every part of the Large System goes in the most probable state during interactions with the other parts.

If we "isolate" some part of the Large System in order to observe the distribution of the probable states, we have to consider the rest of the Large System as a thermostat, that is a reservoir that ensures equilibrium (in a thermodynamic sense) of the distribution of states within the subsystem isolated.

To find out the form of this distribution, it is necessary to introduce the notion of "temperature", equal to the inverse of the derivative of entropy of the Large System with respect to the macro-parameter for which there exist a conservation law.

The introduction of the parameter of equilibrium _temperature_ is simply a result of the condition that _there are no flows of the conserved macro-parameter between the parts of the system_. If there are several conserved macro-parameters, then there are as many parameters of equilibrium as there are conservation laws.

Observe that we said nothing related to either physics, or physical laws and observables. All the arguments are applicable to Large Systems of any nature, subject to the above hypotheses. These arguments look rather natural, in relation to the large economic systems, if we regard the total income, the total value of products or the total value of consumption of goods as macro-parameters, whereas distribution of income and products of consumption of goods between the subjects of economic activity are viewed as micro-parameters.

In this approach, to the study of economic equilibrium, we avoid the necessity to explore the subsystem’s dynamics, so the knowledge of institutional limitations on the goods production and distribution suffices. Thus, we offer

_not only a new approach, appropriate to describe the economic equilibrium, but, also, the instruments to investigate the impact of institutional restrictions on the state of equilibrium._

Namely, we get the opportunity to build up a mathematical apparatus for analysis of the transaction costs theory, to suggest quantitative methods of the study of the impact of the information asymmetry on the behavior of the market's agents and to tackle many other problems as well.

Observe that, for more than a century, the endeavors to prove thermodynamic prediction within the framework of theoretical physics by analyzing equations of motion were not successful.

Nobody doubted that the principles of thermodynamics work, regardless of possible reductionist interpretations. The two levels of description pose the problem: how to single out certain parameters, perhaps, completely “inconspicuous” or not obvious, that govern the condition
for equilibrium, intuitively understood as the absence of significant flows between the parts of the system.

If the macro-parameters are functionally dependent, and the surface of state is differentiable, then the differentials of the macro-parameter produce are related by a system of Pfaffian equations.

Thus, the idea of thermodynamic equilibrium is quite appropriate for description of economical systems. They have observable flows of money, goods and people, and, like physical systems, have two levels of description

Consequently, the description of the economic system should be equivalent to the description of physical systems in thermodynamics, though the parameters of equilibrium _ “temperature”, “pressure”, “chemical potential” _ will certainly have quite different interpretations, the ones that mirror the particularities of economical systems.

Observe that thermodynamic terms had often been already used, by folk, not scientists, in relation to the economical systems in a “naive” manner, as metaphors of description: the stock exchange is “overheated”, the national economic problems “boiled over” or “cooled down”, stock exchange indices are associated with temperature degrees of a thermometer, etc.

One of the aims of this work is, besides all, to show that, not rarely, there is more sense in the “naive” metaphors of such sort, than in the complex mathematical models based on mechanical metaphor of equilibrium. Proceeding from the thermodynamic metaphor, a thermodynamic theory of economics can be developed. It not only catches hold on certain realities of markets by no means less than the one built on the mechanical metaphor, but also accounts for the role of institutional restrictions for the establishment of economic equilibrium _ the task unfeasible to the theories based on the mechanical metaphor.

§2. The entropy and temperature in models of economics

We begin our description of a thermodynamic model of economics with the simplest example. Let the economic system consist of N agents, among whom the income, constant for the system as a whole, is distributed. We assume that there are many ways to distribute income, and we are incapable to foresee all possible alternatives. This assumption fully corresponds to Hayek's concept of market in which the market is described as an arena of discoveries of new procedures and operations. For ultimate simplicity, we assume that the income is quantum, i.e., presented in integers (which is natural, as the smallest unit of currency is operational in economics).

Now, for each value of the total income (E) it is possible to find the quantity of modes of income distribution between the agents, as a characteristic n(E,N) of this value, called the statistical weight of the state with income (E).

At this stage we can introduce the concept of equilibrium. The idea is that two systems (under the above hypothesis) are in equilibrium, if the distribution function of income does not change when they enter in a contact, hence, there is no income flows between the systems. By a “contact” we understand here an “open list” of possible modes of redistribution. It is remarkable that it is possible to calculate the statistical weight regardless of uncountable variety of various institutional limitations imposed on the agents’ incomes, so functions n(E,N) may be different for different systems.

The given model is rather simple: at this phase of reasoning we do not really turn our face to the market. The restrictions on income may be sustained coercively, but the actual means are irrelevant for this investigation.
In thermodynamics, the logarithm of the statistical weight is called the entropy (of the system), and its derivative on energy is inverse temperature,

\[ \frac{\partial}{\partial E} \ln n(E, N) = \frac{1}{T}. \]  

(2.4)

In order to reach the state of equilibrium, the interacting systems should be at the same temperature.
The italicized statement above, together with its deduction, can be found in any textbook on statistical thermodynamics. Let us analyze here the adequacy of such an approach to the economical systems, at least, under the above assumptions. The economical system is in the state of equilibrium if it is rather homogeneous and there are no income flows from one of its part to another. We suppose, certainly, that the homogeneity is kept only unless there is no division into parts so tiny that significant income flows are observed.

The same postulates are available in statistical physics. Subdivision of the system into too small parts results in significant fluctuations. In physics, the question of an equilibrium of small parts of the system is solved by assuming (and this assumption is equivalent to a postulate of equal probability of elementary states), that if we observe a small part of system for "sufficiently long" time, then we will be able to adequately describe the distribution of probabilities of its state. It is one of the formulations of the so-called ergodic hypothesis.

The similar hypothesis can be made for the economic systems. We see that in our thermodynamic model, there are two extremely important characteristics—the entropy and temperature. If we do not know these parameters, we can not correctly infer the conditions for the system's state of equilibrium: as the system is in the state of equilibrium only when its subsystems have an identical temperature, and the temperature cannot be calculated without knowledge of entropy.

In physics, to measure temperature, one uses thermometers. These are special devices whose equations of state are known and calibrated; so by introducing the thermometer into a contact with a body we may find the temperature of the body by a change of the state of the thermometer. As we will see, the (stock) markets may, to an extent, be considered as thermometers in economics.

The basic possible objection against the thermodynamic approach to economy is that the number of "particles" involved (in the given example—the number of the market agents) is much less as compared with numbers of particles in the usually considered physical systems. In the physical systems, the number of particles is comparable, as a rule, with the Avogadro number, whereas in the economic systems it is usually is \[ N \approx 10^3 - 10^8 \].

In statistics, the order of dispersion is equal to \[ \frac{1}{\sqrt{N}} \], where \( N \) is the number of particles in the system. Thus, in physical problems, if the statistical errors related with the fact that the number of particles is finite are completely insignificant due to the largeness of their number, in economic problems we should expect much larger errors, \[ \% 3 \% \], or less. Such errors do not look too large ones, actually, taking into account the extreme roughness of economical models.

One of the tendencies in contemporary physics is to apply thermodynamic approach to systems with a rather small number of particles \[ 10^3 - 10^8 \] (nuclear physics, cluster physics, etc.) and the results appear to be quite valid not only qualitatively, but also quantitatively.

It seems that both models of interaction and the data on interactions are far from being accurate, and to strive for a better accuracy in meaningless: there are in-built limits of accuracy, like the uncertainty principle. The situation with applicability of thermodynamic approach to economics seems to be similar.

An important remark. The statistical models in physics show that the energy of the system has a peculiar quality responsible for a success of the thermodynamic theory: if the system has some parameter of inhomogeneity, then, provided there are sufficiently many particles, the system has a very sharp maximum of entropy attained in the limit as the parameter of inhomogeneity tends to zero. Thus, not only the maximally homogeneous state is most probable, but even small deviations from it are most improbable.
To illustrate this thesis, let us analyze the entropy of one very simple system. Suppose that the income distribution between the subjects of economic activity is arranged as follows: each subject has either zero income, or a fixed income, $A$ assuming that the system is organized in such a way that all deviations are annihilated through special institutional redistribution mechanism.

Such an example is not so much unreal, if we recall the economic experience of some countries aspiring to implement various "leveling principles" in distribution of income, stipulating that a part of population is totally excluded from economic activities, being allowed to have a very low level of income (subsidized by a social security or by Nature). If the total number of economic subjects is $N$, the number of the subjects with income $A$ is $L$ (hence, the size of the total income is $E_{tot} = LA$), then, clearly, number of probable states of this system would be equal to $C_N^L$.

By comparing the statistical weights of the systems with the different levels of total income, we conclude that the biggest statistical weight is the characteristic of the state with $E = LA$, where $L = N/2$ (to simplify arguments, let $N$ be even). Introducing a parameter of inhomogeneity, $m = [L - N/2]$, we apply the Stirling formula for the factorial. We obtain the following simple approximation to the dependence of statistical weight on the parameter of inhomogeneity:

$$n(N, m) = 2^N \sqrt{\frac{2}{\pi N}} \frac{N}{2} \exp \left( \frac{2m^2}{N} \right).$$  (2.5)

Formula (2.5) shows that the statistical weight (and, consequently, the entropy) has an extremely sharp maximum depending on the parameter of inhomogeneity, with width $\frac{1}{\sqrt{N}}$.

The principle of equal probability of elementary states immediately implies that the entropy of interacting systems tends to increase. The most probable state is the state with the greatest statistical weight, i.e., the state with the maximum entropy. As the number of probable system’s states sharply decreases with the increase of the parameter of inhomogeneity, it is hardly possible to find the system in a state for which the parameter of inhomogeneity exceeds a certain value determined by the number of particles in the system (in the example above, this value is $\frac{1}{\sqrt{N}}$).

§3 The thermostat and the function of income distribution

Now we will investigate how the income is distributed among the subjects of economic activity. Within the framework of thermodynamic model, it appears that there exist a universal function of dependence of statistical weight on the parameter of inhomogeneity:

$$P_1(E_1) = \frac{\eta(E \otimes E_1, N)}{\eta(E \otimes E_2, N)}.$$  (3.1)
Since \( n = e^{S(E, N)} \), where \( S(E, N) \) denotes the entropy of the system, we can express as:

\[
\frac{P(E_1)}{P(E_2)} = e^{S(E_1, N) - S(E_2, N)} = e^{S}
\]

(3.2)

If the reservoir is far greater than the subsystem in question, we can make expansion of \( S \) into Taylor series in \( E \), thus obtaining in the first order \( S = \frac{E_1}{T} \), where \( T \) is the temperature of the reservoir, see (2.4).

It follows that:

\[
\frac{P(E_1)}{P(E_2)} = e^{\frac{E_1 - E_2}{T}}.
\]

(3.3)

This is an approximate formula for the probability distribution for a small system.

If the number of probable states of subsystem \( Y \) with incomes \( E_1 \) and \( E_2 \) is equal to \( n_Y(E_1) \) and \( n_Y(E_2) \), respectively, then the distribution formula takes the form:

\[
\frac{W_Y(E_1)}{W_Y(E_2)} = \frac{n_Y(E_1)}{n_Y(E_2)} e^{\frac{E_1 - E_2}{T}}.
\]

(3.4)

In this formula we proceed from the probability of the state, \( P \), to the probability of the level of income, \( W \). Observe that no assumptions whatsoever were made, except two: the system is homogeneous, so it is possible to subdivide it into interacting parts without producing income flows from one part to another, and the total system’s income is a constant.

The arguments laid out above, are, actually, a replica of the traditional deduction of Boltzmann distribution in statistical thermodynamics.

The Boltzmann distribution function is realizable, with respect to the income, in the system, which interacts with the reservoir kept at certain temperature. In the isolated system, on the contrary, the temperature depends on both income and entropy.

The Boltzmann distribution of income allows one to understand the relationship between the average income of the subjects inside the system, and the system's temperature. Suppose that there are no restrictions on the agent’s income and all levels of income are allowed. Such a situation is commonly associated with the market economy. Then, with the help of the Boltzmann distribution function, we easily calculate the average income of the agent:

\[
E = \int_0^\infty e^{\frac{E}{T}} dE = T.
\]

(3.5)

We see that in absence of any restriction on the agent’s income, the mean income is equal to the temperature. The upper limit of integration here is equal to \( \infty \), despite the fact that even the total income of the real system is bounded. However, since the exponent steeply decrease, this does not matter, provided the total income greatly surpasses the average income of a single agent; in the real systems this is true, of course.

As we will see, under restrictions on income the relationship between the temperature and the average income may look quite different.

In conclusion of this section, observe that, like in statistical thermodynamics, in the thermodynamic model of economics, the parameter called “ statistical sum”,


is extremely useful. The sum here runs over all possible values of \( \epsilon \). The probability of the system interacting with thermostat with temperature \( T \), can be expressed as

\[
\rho(E) = \frac{e^{\frac{E}{kT}}}{Z_0}.
\]  

(3.7)

With a fixed number of agents, the average income in the system at temperature \( T \) is given by the formula:

\[
\mathbb{E} = \frac{\sum_{\epsilon} E n(E,N) e^{\frac{E}{kT}}}{Z_0} = T \frac{\partial}{\partial T} \ln Z,
\]  

(3.8)

This means that if we know how the statistical sum depends on the temperature, it is possible to obtain the value of the average income by differentiation. Again, this is fully agreeable with the standard technique of statistical thermodynamics.

Consider now one paradoxical example. It demonstrates that application of the thermodynamic approach to economic systems helps to deduce unexpected, though true, conclusions.

§4 On interaction of systems with and without restrictions on income

In this section the “spin model”, with two possible values of income, 0 and \( A \), already addressed above, will be examined in more detail. Despite its seemingly too abstract nature, this model is rather useful, as it catches some important features of systems with restrictions on income.

What the results would be once such a spin system enters into interaction with a “free” market system, the one without any restrictions on income? In the free market system, the entropy increases together with the increase of energy, and the temperature is always positive. This is not always the case for system with restrictions on income. Actually, if the number of agents \( L \) with non-zero income surpasses the half of a total number of agents, \( N \), the number \( C_N^L \) of probable states of the system, starts to diminish, and the inverse temperature given by formula (2.4) becomes negative.

What is the meaning of the negative temperature?

The phenomenon was investigated quite well in the laser theory\( ^{xii} \). The negative temperature implies the inverted population, i.e., a situation, in which the levels with higher energy are more densely populated than the ones with lower energy. In such a state, the system is imminently ready to release its energy at the contact with any system with positive temperature.

How can this be interpreted in terms of the income distribution described above? If two systems with positive temperature are in contact, the redistribution of income goes from the system with higher temperature to the one with the lower one. As we saw, for the free market, the temperature is equal to the average income per agent. The redistribution of income from the “richer” system to the “poorer” one takes place in accordance with our intuitive understanding.

Contrariwise, if a free market system interacts with a system with restriction on income, then, under certain conditions, a counter-intuitive process is observed, when redistribution the income goes from the “poorer” system to the “richer” one.
Let \( Y \) be a free market system, with the temperature \( T_X \) equal to the mean income per market agent. Let \( Y \) is a spin model with income bounded from above by \( Y \). Let \( T_X > NA \) and let the number of \( Y \)'s subjects with non-zero income exceed the half of their total number, so the temperature of \( Y \) is, evidently, negative.

What is going to happen when these systems start to interact? The entropy of the joint system will increase, and the entropy of system \( Y \) will also increase simultaneously, thereby precipitating disorder of system \( Y \). Therefore, \( Y \) must transfer a part of its income to \( X \) despite the fact that average income of the system \( X \) is already higher than the average income in \( Y \).

As a result, system \( Y \), with restriction on income, will get still poorer, while system \( X \), without any restriction on income, will get richer and richer.

This redistribution of income will not be an effect of coercion or looting, but just a consequence of the fact that the combined system tends to acquire the most probable state. As for coercion, it may play a certain role, but not in the form of \( X \) raping \( Y \), but coercion inside system \( Y \), aimed at obstructing any rise of income above certain level.

Applying the above most simple model to real economic situations, we can draw two important conclusions:

1) the policy of restriction of income is dangerous if the contact with a free market surroundings is unavoidable;
2) if such a policy was already embarked on, and the economy in which it was implemented is isolated from the free market, then the effect of the “market reforms” will depend on the sequence of two main steps, the release of incomes and the establishment of contact with the free market surroundings.

For the country with a “non-market” economy, it would be necessary first to release the income from restrictions and then to “open” the economy to the market only after the equilibrium had been achieved. Otherwise, the resources of the country will be “sucked out” outward before the equilibrium is established.

Of course, our arguments are based on a very simple idealized model. Still, our reasoning gives an explanation of different results observed at the carrying out market reforms in Eastern Europe, on the one hand, and in China and Vietnam, on the other.

In the Eastern Europe, the reforms were conducted by means of a “shock therapy”: the national economies were “opened up” for outside activities without preliminary creation of market institutions inside the country.

The results were catastrophic (flight of capital and collapse of production).

In China and Vietnam the reverse order was implemented: first, the release of capitals inside the country and then the market was gradually opened up for outsiders, in accordance, more or less, with the process of creation of a domestic market. Such a policy resulted in an astoundingly rapid economic growth.

Consider another case when a system with a restriction on income interacts with a free market system.

Let the “spin system”, a community of \( N \) workers, with fixed individual income \( A \), have \( L \) vacancies. Obviously, for \( A \) fixed, the system’s temperature is negative for \( L > \frac{N}{2} \).

This means that in the state of equilibrium, the free market would prefer that half of the workers should not get salary, i.e., should be unemployed, because this is a far more probable state than the other ones with a lower rate of employment. So, this model includes unemployment as an inherent characteristic of the system in the state of equilibrium.
For the neo-classical school of economics, the following famous paradox stood as a stumbling block: why the market equilibrium is not realized when the free labor force market is in operation? In the above model this paradox is dismissed.

A negative temperature may also emerge in the system with restrictions on the salary's from above. Indeed, to increase the total value of wages in the situation of a salary-restricted economy with the value of salary bounded from above by $A$, means that at the total value of wages $E$ equals $NA$, the entropy vanishes. This implies that, at least for some interval $E_0 < E < NA$, the temperature, $T = \frac{1}{\partial S / \partial E}$, is negative, i.e., at any contact with a free market the value of total wages will fall, at least to $E_0$.

If, moreover, there is a minimal wage value, $B$, the number of gainfully employed will be bounded from above: $L < \frac{E_0}{B}$.

This restriction will keep salary from dropping down, and necessitate a certain rate of unemployment, which will be $\geq \frac{N}{N} L = \frac{N[1]}{N}$. In reality, the level of wages is always limited “from below” by the level of biological survival (at least, in towns, where Nature's resources are unattainable). Therefore, if salary is bounded, the opportunity for unemployment is created.

The analysis given above shows that unemployment emerges in the state of equilibrium at the positive temperature if the wages are bounded simultaneously from above and below.

The above can be considered as a proof of Keynesian arguments on the reasons of unemployment. In real situations, the administration aspires to put a limit on the salary “from above”, while the trade unions (are supposed to) do it “from below”. This combination must, according to our model, result in the emergence of unemployment.

§5 The migration potential

In the previous section we showed some applications of the thermodynamic approach to economics and interpreted in terms of economics several thermodynamic parameters.

One more very important parameter is known in statistical physics under the name the chemical potential. Here it will be referred as migration potential. It describes the state of equilibrium in the systems when the number of agents is not fixed.

Take two such systems, assuming that the total number of their agents is a constant, $N_1 + N_2 = N$, but the agents can migrate from one system to another. Consider the problem: under what conditions these two systems will get into an equilibrium?

Having applied standard technique, it is possible to find out when the entropy of two interacting systems, characterized by restrictions on the total amount of income and the total number of agents, will be maximal. The total entropy is the sum of entropies of the two systems. As a result, we get two conditions (as a corollary of $dN_1 = [dN_2$ and $dE_1 = [dE_2$ implied by $dN = 0$ and $dE = 0$):

$$\frac{\partial S_1(N_1, E_1)}{\partial N_1} \bigg|_{E_1} = \frac{\partial S_2(N_2, E_2)}{\partial N_2} \bigg|_{E_2}; \quad (5.1)$$
\[
\frac{\partial S_1(N_1,E_1)}{\partial E_1}
\bigg|_{N_1} = \frac{\partial S_2(N_2,E_2)}{\partial E_2}
\bigg|_{N_2}.
\]

(5.2).

The factor \( \Box = \int T \frac{\partial S(N,E)}{\partial N} \bigg|_E \) will be called the \textit{migration potential}. When the temperatures are equal, the two systems in diffusional contact, i.e., when migration of agents from one system to another is allowed, are in equilibrium (i.e., in the most probable state), if their migration potentials are equal.

As \( \mathcal{N}_1 \) agents from the second system migrate to the first one, the change of entropy will be as follows:

\[
\Box S = \Box S_1 + \Box S_2 = \frac{\Box S_1}{\mathcal{N}_1} \bigg|_{\mathcal{N}_1} + \frac{\Box S_2}{\mathcal{N}_2} \bigg|_{\mathcal{N}_2} = \frac{\Box N_2}{T} \frac{\Box N_1}{T} \mathcal{N}_1
\]

(since \( \Box N_2 = \Box \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_1 > 0 \)).

This makes clear what happens with the system, if migration potentials are not equal. If \( \Box_2 < \Box_1 \), the change of entropy with the increase of number of the agents in the first system is positive, hence, it is a change of state towards the maximum probability.

In other words, the agent flow proceeds from the system with a larger migration potential to the system with a smaller migration potential.

Now, we can find the relative probability of various states for the system interacting with a thermostat and exchanging with it either income, or agents, or both. In the same way as before, we obtain:

\[
\frac{P_y(E_1,N_1)}{P_y(E_2,N_2)} = \exp\left\{S(E_1 \Box E_2, N \Box N_1) - S(E \Box E_2, N \Box N_2)\right\},
\]

(5.4)

where \( (E_1,N_1) \) and \( (E_2,N_2) \) are different states of subsystem \( Y \), interacting with the thermostat, while \( N \) and \( N \) are the total income and the total number of agents, respectively, in the total system (sub-system \( Y \) plus the thermostat).

On expansion into Taylor series and simplifications, we obtain:

\[
\frac{P_y(E_1,N_1)}{P_y(E_2,N_2)} = \frac{\exp\left\{(N_1 \Box E_1)/T\right\}}{\exp\left\{(N_2 \Box E_2)/T\right\}}.
\]

(5.5)

This means that the probability for subsystem \( Y \) to exchange income and agents with thermostat is proportional to the Gibbs’s factor

\[
\exp\left\{(N \Box E)/T\right\}.
\]

As it was mentioned earlier, the thermostat is just the remaining part of the large system, out of which the subsystem \( Y \) was isolated. As other thermodynamic parameters, we obtain the Gibbs’s factor in a way usual for statistical physics.

A question arises: \textit{are the hypotheses on the total system’s income and the number of agents too binding and non-realistic?}

It looks that hypotheses of the kind are always needed in the study of the properties of ideal systems. We observe very similar problems in statistical physics as well. There, certainly, no absolutely closed system can be found, still, nevertheless, the theory "works".

In real economic systems, the number of agents is, of course, much smaller than that in physical systems, but, on the other hand, the expected accuracy of prediction is not so high, either.
The Gibbs’s factor paves the way for introducing a very useful parameter, called in statistical thermodynamics the large statistical sum:

\[
\mathcal{Z} = \sum_{N=0}^{\infty} \sum_k \exp\left(\frac{\left\{ N \square E_k(N) \right\}}{T} \right). \quad (5.6)
\]

The large statistical sum allows one to compute, effortlessly, some important parameters of the system. For instance, the mean number of the system’s agents can be found by differentiation with respect to \( \mathcal{Z} \).

Set \( \square = e^{\mathcal{Z} / T} \). Then the large statistical sum takes the form:

\[
\mathcal{Z} = \sum_{N} \sum_{k} e^{N} e^{\mathcal{Z} / T}. \quad (5.7)
\]

Now it is easy to deduce (see [IV]) that the mean number of agents is equal to:

\[
\bar{N} = \frac{\partial}{\partial \square} \ln \mathcal{Z}. \quad (5.8)
\]

This relation is very important. It provides with a way to determine \( \square \) in the investigated systems by equating the number of agents in the system to \( \langle \bar{N} \rangle \).

Here is an example from the economics of migration, aimed to demonstrate the usage of the migration potential. Consider two systems: _A_, with \( N_\text{A} \) many vacancies, and _B_, with \( N_\text{B} \) many vacancies, with the total of \( n \) agents capable to freely migrate between the two systems. Let the per capita income in _A_ be \( \bar{A}_\text{A} \), that in _B_ be \( \bar{A}_\text{B} \). Besides, we assume that both systems are immersed in a much greater system, a thermostat with temperature \( T \).

How do probabilities of filling vacancies in systems \( A \) and \( B \) depend on the parameters of the model? The equilibrium, considered here as the most probable state of the joint system, will be determined by the system's temperature and migration potential. Using the large statistical sum, one easily obtains the function of distribution of agents for \( A \) and \( B \), see\textsuperscript{xiv}.

Let one vacancy in system \( A \) be isolated out as a subsystem. Assuming that this subsystem is in equilibrium with the remaining part of the system, it is possible to determine the large statistical sum of the subsystem. As this subsystem can only be in one of the two possible states, with income 0 (the empty state) and with income \( \bar{A}_\text{A} \), the large statistical sum is equal to:

\[
Z_\text{A} = 1 + \square e^{\mathcal{Z}_\text{A}} / T, \quad \text{where } \square = e^{\mathcal{Z} / T}. \quad (5.9)
\]

Similarly, for system \( B \):

\[
Z_\text{B} = 1 + \square e^{\mathcal{Z}_\text{B}} / T. \quad (5.10)
\]

As systems \( A \) and \( B \) are supposed to be in equilibrium, and, therefore, their migration potentials are equal and \( \square \) is the same for both systems. Hence, the probability of one vacancy in system \( A \) to be occupied is equal to:

\[
\square_\text{A} = \frac{\square e^{\mathcal{Z}_\text{A}} / T}{1 + \square e^{\mathcal{Z}_\text{A}} / T}. \quad (5.11)
\]

Similarly, for system \( B \):
\[
\square_B = \frac{E_B}{1+E_B^T}, 
\]
and \( \square_A/\square_B \) is equal to:

\[
\frac{\square_A}{\square_B} = \frac{E_A}{E_B} + \theta. 
\]

Since the total number of the system’s agents is equal to \( n \), we have:

\[
n = N_A \square_A + N_B \square_B. 
\]

We determine the migration potential from this equation; we substitute it into the expressions for \( \square_A/\square_B \) thus finding the probabilities for agents to belong to \( A \) and \( B \), respectively, in the equilibrium state. It is clear that as the temperature of the thermostat changes, the migration potential also changes, because the relative probability for the joint system \( + \) to have total income \( \_ \) depends on the Boltzmann's factor \( e^{-E_j/T} \).

Now, consider the case when the numbers of vacancies in the systems in question are equal and both of them are almost totally occupied, i.e., both \( \square_A \) and \( \square_B \) are close to 1. This means, that we can expand \( \square_A \) and \( \square_B \) with respect to \( \square^E_A \) and \( \square^E_B \). We will confine ourselves to the first order terms. Simple calculations result in:

\[
\frac{\square_A}{\square_B} = \frac{E_A}{E_B} + \frac{1}{N} + \frac{1}{N}, 
\]

where \( N = N_A - N_B \) is number of vacancies, \( \square = 2N \square_n \) is total number of vacancies in the joint system, and \( x = e^{-E_j/T} \).

Clearly, if \( E_A - E_B \) is far greater than the temperature, then the ratio of probabilities becomes independent of the income, and tends to \( 1/N \). If \( E_A - E_B \) is small, we have

\[
\frac{E_A - E_B}{T} \sqrt{1 + \frac{E_A E_B}{T}} \text{ and } \frac{\square_A}{\square_B} \left( \frac{E_A E_B}{2TN} \right). 
\]

Thus, not the values of \( E_A \) and \( E_B \) are essential, but the ratio \( \frac{E_A E_B}{T} \), the parameter determined by the thermostat, that is by environment.

Assuming that the thermostat is a free market system, and, therefore, the temperature in it is equal to the mean income, it becomes manifest that to find the relative probability of employment in two systems, the ratio of the difference of wages to the mean wage in the environment is essential.

So far we had considered very simple model problems of employment, in order to demonstrate a possible way of solving this kind of problems in general. Clearly, one can consider more realistic
assumptions, e.g., the investigate migration processes in interacting systems characterized by distinct restriction of income.

Such models are of particular importance in the study of regional economies.

§6 The thermodynamics of the prices: a description of the problem

Let's try now to extend the thermodynamic approach to the study of the markets and prices. For this purpose, consider a simplified model situation. It is necessary, nevertheless, to simplify with caution, so as not to lose the most essential situational characteristics.

Our first model is intended to analyze how the size of flow of the goods and money influences the market prices.

In the above section we showed that if the money flow is constant, it is possible to introduce the concept of equilibrium in the income distribution. To implement this, we have to know the entropy of the system, hence its temperature. Using temperature we are able to find out the conditions under which the system stays in equilibrium with the environment.

Now we add to this model a flow of goods. In what follows we will discuss how this additional parameter can be interpreted in terms of the price of the goods. Moreover, it is also possible to deduce the equation of the market state, i.e., to find dependence between the flow of goods, the price, the number of buyers and temperature.

First, consider an intuitive concept of the market and the market equilibrium, and then try to formalize it.

The market is characterized by the presence of goods which are sold, and money which are spent at purchases. Let \( V(t) \) be the amount of “units of goods” sold per unit of time during which the buyers spend \( E(t) \) of units of money. We say that the market is stationary if \( V(t) \) and \( E(t) \) are constants independent of time and all goods are bought, i.e., there is no accumulation of goods in the hands of the sellers. While the market functions, the deals are made, i.e., agreements on exchange of some of the goods for some money.

Bargaining regulations are typical for the market. Some kinds of deals may be ruled out, for instance, the ones bidding price too high or too low. What is important for us is that the regulations on deals do not vary with time.

Observe that even if the price regulations are not officially fixed, some rules exist anyway, e.g., the ones that ensure contracts' fulfillment. Thus, the market is a social institute due to the existence of rules of dealing. No doubt, the rules imply that a control apparatus should be present, its purpose to enforce the rules. It has to be entitled with certain enforcement powers, in order to punish violators and guarantee reliability of contracts. In this sense, the market is certainly not the arena for totally spontaneous activity of its agents, but an organized social institute.

Consider a model with several co-existing markets able to interact: exchange goods and money resources. The intuitive concept of equilibrium of these markets is that the situation in each of these markets remains, in certain essential aspects, the same even after the interaction. Clearly, this cannot take place for any values of the amounts of goods and money \( V_j, E_j \) and \( V_k, E_k \), in each system, respectively. Besides, the interaction can vary, i.e., include an exchange only of money resources, only of goods, or both.

Consider the simplest case, when the values of money and goods are discrete, as is the case in real life:

\[
E_n = nE_0, \quad V_m = mW. \]  \hspace{1cm} (6.1)
In addition, let the values of \( n \) and \( m \) be sufficiently large, to make it possible to treat small changes of flows as insignificant and differentiate. Let \( N \) be the number of buyers in our model. (Generally speaking, \( N \) should be the number of bargains, but it is more convenient to consider \( N \) as the number of buyers). The market is assumed to be stationary in the above sense.

The set of probable state of the market is the set of all contracts, allowed by rules, i.e., the ways of distribution of the goods between buyers. Let us examine the market with a single seller assuming that the seller is not capable to influence the flow of arriving goods.

For this model, introduce the entropy in the same way as earlier, namely, as the logarithm of the number of probable states. In this case, the entropy depends on both the total money expenditure, \( E \), and the total amount of purchased goods, \( V \). Various states of the market can be regarded as legalized by the rules of distribution of the total supply of money and goods among \( N \) buyers. Observe that now we study the “market of the buyers”, so, we do not have to include in the entropy the distributions of goods among sellers.

If, however, we do take them into consideration, we have to incorporate in the model the distribution of the money among the sellers. This is a far more complex task.

To determine the price, it does not matter where the buyer purchased the goods. What is important, is how many goods he or she has received and how much he or she has paid.

So, for the analysis of the “buyer’s market”, the entropy, due to the presence of many sellers at the market, is not considered.

To avoid all this complications, we could have counted the number of contracts, instead of the numbers of sellers and buyers. Such an assumption, however, requires to introduce the migration potential corresponding to possible change of the number of contracts. This case will be considered later.

By analogy with the pressure in statistical thermodynamics, the notion of marginal price \( P \) is introduced as follows:

$$ P = T \frac{\partial S}{\partial V}. \quad (6.2) $$

A bit later we will justify the introduction of this parameter. The marginal price is a characteristic that points out whether or not the two interacting systems are in the equilibrium.

First, consider the simplest case, referred to as the “free market”. In this model there are no restrictions on goods and money distributions among the buyers. This means that the buyer can pay any price – either infinitesimally low or infinitely high.

The free market is rather easy to study, because the total number of probable market states is the product of the number of possible distributions of goods between the market agents times the number of possible distribution of money between the same agents. Thus, the statistical weights of the system are obtained by multiplying the statistical weights determined by the flows of goods \( (V) \) and money \( (E) \). Consequently, the entropy of the system (the logarithm of the statistical weight) consists of the two components: one, is determined by the flow of goods, the another one, by the flow of money:

$$ S(E,V) = S(E) + S(V). \quad (6.3) $$

Let the money and goods flows be quantized, as in (6.1). In what follows we will derive the equation of state for such a system.

First, compute the number of probable distributions of goods and money flows among \( N \) agents of the market. The statistical weight \( g(E_n,N) \) of the flow \( n \), distributed among \( N \) agents is equal to the number of non-negative solutions of the equation:

$$ n = x_1 + ... + x_N. \quad (6.4) $$

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The calculational technique for such equations is well-known. Namely, add 1 to each \( x_i \). Then the quantity to be determined has is the number of positive integer solutions of the equation:

\[
n + N = y_1 + \ldots + y_N.
\]

(6.5)

Now, let us subdivide the integer segment of length \( n + N \) into \( N \) integer subsegments. Thus, the statistical weight is equal to:

\[
g(E_n, N) = C_{n+N}^{N!} = \frac{(n + N)!}{(N!1)!n!}.
\]

(6.6)

For sufficiently large \( n \) and \( N \) the Stirling formula gives:

\[
g(E_n, N) \approx \frac{1}{\sqrt{2\pi}} \frac{(N + n1)!}{N!1!n!}. \quad (6.7)
\]

Hence,

\[
S(E_n) = \ln g(E_n, N) \approx (N + n1)! \ln n + \frac{1}{2} \ln n + \frac{1}{2} \ln 2 \pi.
\]

(6.8)

The temperature \( T \) is calculated in terms of \( \frac{\partial S}{\partial E} \):

\[
\frac{1}{T} = \frac{1}{\sqrt{n}} \frac{\partial S(E_n)}{\partial n} \ln n + \frac{1}{2} \ln \frac{n + 1}{2}.
\]

(6.9)

For \( n \approx N \), which is a rather natural condition for the free market, the expression for the inverse temperature is:

\[
\frac{1}{T} = \frac{\partial S(E_n)}{\partial n} = \frac{1}{n} \ln n + \frac{N}{E}.
\]

(6.10)

This means that the temperature \( T = \frac{E}{N} \) is equal to the mean value of income per capita.

To compute \( \frac{\partial S}{\partial V} \) is a completely analogous matter (since the entropy \( S \) can be described as the sum of two terms, one of which only depends on \( E \), the other one, on \( V \), see (6.3)):

\[
P = \frac{\partial S(V_m)}{\partial V_m} = \frac{1}{Wm} = \frac{n}{V}.
\]

(6.11)

Thus, a relation between the marginal price, the goods flow and the "temperature" of the free market is of the form:

\[
P = \frac{T}{V}.
\]

(6.12)

As \( TN = E \), it is easy to show that for the free market the marginal price is equal to a median price. This, actually, allows to define the price as \( T \frac{\partial S}{\partial V} \).

In presence of restrictions on price, the marginal price may deviate from the median one.

Manifestly, the equation of the free market (6.12) is totally analogous to the equation of the ideal gas.
It might seem that we again reproduced the arguments from a textbook on statistical physics, having changes semantics. This not so! Though the marginal price does coincide, in its form, with the expression for the pressure in statistical thermodynamics \( P = T \frac{\partial S}{\partial V} \), this expression is not the definition of the pressure. The origins of pressure are of different, mechanical, nature and the pressure is defined to be equal to \( \partial E / \partial V \).

In our arguments, the marginal price has the same thermodynamic origin as the temperature or the chemical potential, i.e., the theory is completely derived from thermodynamic principles. The marginal price is the parameter of equilibrium, identical to the temperature. It indicates whether or not the interacting systems are in the equilibrium.

Indeed, the same pattern of reasoning as at the introduction of the temperature, can be repeated and we similarly prove that two interacting systems for which the exchange of the goods is possible are capable to be in the state of equilibrium only when their marginal prices for the goods are equal.

Indeed, let the statistical weights of the interacting systems depend on the flows of money and goods, \( E \) and \( V \), while the total values of both parameters are constants (speaking about a flow we mean its expenditure capacity per unit time):

\[
E_1 + E_2 = E, \quad V_1 + V_2 = V,
\]

then the statistical weight of the combined system is:

\[
g_1(E_1,V_1)g_2(E_1,V_1,V_1).
\]

The probability of the state becomes maximum when the function (6.14) attains its maximum.

Introducing entropy as \( \ln g \), the conditions for the entropy extremum become:

\[
\begin{align*}
\frac{\partial S_1}{\partial E_1} dE_1 + \frac{\partial S_1}{\partial V_1} dV_1 + \frac{\partial S_2}{\partial E_2} dE_2 + \frac{\partial S_2}{\partial V_2} dV_2 &= 0.
\end{align*}
\]

Taking into account that

\[
dE_1 = \Box dE_2, \quad dV_1 = \Box dV_2,
\]

we express the conditions of equilibrium as follows:

\[
\begin{align*}
\frac{\partial S_1}{\partial E_1}|_{E_2} &= \frac{\partial S_2}{\partial E_2}|_{E_2}, \quad \frac{\partial S_1}{\partial V_1}|_{E_2} = \frac{\partial S_2}{\partial V_2}|_{E_2}.
\end{align*}
\]

The first expression, as we already discussed, means the coincidence of “temperature”. The second one means coincidence of values of the marginal prices.

Thus, coincidence of values of the marginal prices is a necessary condition for the interacting markets to be in the state of equilibrium.

Now, proceed to the case that the market is not “free”; let some restrictions be imposed on its operation.

Let us investigate, first, what is going to happen if something similar to the “allocation system” is adopted, i.e., the possibilities to buy goods are forcefully restricted. Recall also our assumption that the market is stationary, which means that the flows of goods and money are completely distributed among the market participants.

Suppose that none of the market agents is allowed to get more goods than a certain amount, \( K \). In all the other respects the distribution remains free, and _ this is of utter importance _ it does not depend on the money distribution, i.e., there are no explicit restrictions on the prices. This means, that the statistical weight of the state with \( V \) goods, and _ money is still the product of statistical weights:
\[ g_n(E,V) = g_n(E)g_n(V), \]
while the entropy includes two components, one of which depends on \( E \), and the other one on \( V \):
\[ S_n(E,V) = S_n(E) + S_n(V). \] (6.17)

To determine the marginal price, we should known \[ \frac{\partial S}{\partial V}_{E,N} \], i.e., we should compute \( S_n(V) \). This
appears to be a difficult combinatorial problem: find the number of non-negative integer solutions of
the equation:
\[ m = x_1 + x_2 K \bigcup x_3, \] where \( x_i \bigcup K \) for each \( i \). (6.18)
The solution of this problem is distinguished by a special feature, which renders meaningful conclusions.

It is clear that if \( m > KN \) this problem has no solutions at all. The logarithm of the number of
solutions is the system's entropy \( S^K_n(V_m) \). This means that when \( m \) (i.e., the flow of goods)
approaches \( KN \) from below, then at some point the entropy begins to decrease. But this, in turn,
means that \[ \frac{\partial S(V_m)}{\partial V} \] becomes negative. Taking into account that the distribution of money flow is
free, which implies positive temperature, the result would be that, under restrictions, the system's
marginal price becomes negative when the volume of the flow of goods becomes sufficiently large.

This means that during interaction with other systems with positive marginal price, our
"allocation system" with imposed restriction on consumption of the goods will begin to eject goods
at dumping prices: any more-than-zero price of the exported goods will contribute to stabilization
of the equilibrium.

Of course, it is possible to say that when the flow of goods is close to the value of \( KN \), there is
no sense to restrict the distribution. The fact, however, is that the marginal price can become
negative long before the flow of goods approaches the critical value, \( KN \).

For instance, for \( K = 1 \), this will happen for \( m > \frac{N}{2} \).

Since the introduction of restrictions such as in our example is a political decision, the question
should be addressed: **how the persons responsible to make such decision could learn that the
marginal price has already become negative?**

Indeed, the price of market transactions is positive, and the median price is also positive. In
order to know how far the system that interact with other systems is from the state of the
equilibrium, it is necessary to calculate certain, not directly observable, parameters _ the
temperature and the derivative of the entropy with respect to the flow of goods.

As we have just shown, this problem is rather tough, even for a very simple model.

We see, nevertheless, that the restrictions imposed on the market can create, in the range of the
system's states, a zone of "latent instability".

Let us consider now whether the zones of latent instability would emerge if the price is bounded
from below. This model is very important indeed because real market systems are hardly free: it is
not realistic to sell the goods, on the large scale, at the price below the production cost.
If in our model the price of the bargain is bounded from below, then the entropy of the system
cannot be described any more as the sum of two components, each depending on only one variable,
one being the entropy of the flow of goods, the other one the entropy of the money flow.
The mathematical problem of computing the entropy is to determine $\ln g_n(E, V_m, n)$, where $n$ is a parameter bounding the prices from below, and $g_n = \prod g(y_i K y_n)$, where $g_n(y_i K y_n)$ is the number of positive integer solutions of the equation:

$$m = x_1(y_1) + x_2(y_2) + K + x_n(y_n),$$

(6.19)

In (6.19) $y_n$ is an arbitrary partition of the number $n$ into $N$ non-negative integers, and

$$x_i(y_i) = 0 \text{ for } y_i = 0 \text{ and } x_i(y_i) < \frac{y_i}{\prod} \text{ for } y_i > 0.$$

(Here $x$ is the allowed number of goods in a deal while $y$ is the corresponding amount of money).

This is a still more difficult calculation problem. But, as in the previous case, some qualitative dependencies can be discovered relatively easy. If we increase the goods flow but the money flow does not increase simultaneously, the problem has no solution. Indeed any possible price remains below the allowed level once $V_{k,m}$ is great enough for limitations on $x$, where the $y_i$ are $\prod n$ and $n$ is such that $\prod = \prod$, i.e., the entropy $S_0(E, V)$ vanishes.

Like in the previously given model, the marginal price becomes negative earlier than that because the distribution of the money flow is free, the temperature is positive, but the derivative $\frac{\partial S_0(E, V)}{\partial V}$ at values of $V$ smaller than but close to $V_k$ becomes negative.

So, we observe the same effect as under the terms of the restrictions on goods, namely, the latent zone of instability with a negative marginal price, and the same consequences _ an outward dumping of goods by the system striving to reach equilibrium.

Apparently, this explains phenomena of mass destruction of the goods for the sake of maintaining the level of prices during the crises of overproduction. It is not the "malicious will" of owners of the commodities, but simply the shortest way to the most probable state of the system: getting rid of the goods the system assumes a statistical equilibrium, i.e., a most probable state.

As a byproduct of our arguments, we see that introducing restrictions in the system we provoke occurrences of zones of latent instability. The borders of these zones are extremely scarcely discernible even for the simplest models.

Such parameters, as derivatives of the entropy, e.g., the temperature and the marginal price, are crucial for the description of the behavior of systems with restrictions after they became engaged in interaction with similar systems or with free-market systems.

The marginal price becomes a very important parameter, a major parameter of the market equilibrium. It coincides with the median price only for the free market systems. In the systems with restrictions, it may be negative. If this happens, the system is unstable.

Thus, restrictions on economic activity can, by no means, be held as "harmless". First of all, no restriction is harmless because the range of their influence is unclear, the range within which their influence renders the system unstable.

Observe that the so-called "economic" reasons for various restrictions on deals (e.g., the minimal price determined by the cost of production) turn out, at deeper scrutiny, purely political reasons: the cost of production is often can not to be lowered "thanks" to a monopoly, a political control of the market.

In other words, the restrictions imposed on the freedom of market activity, are capable to generate uncertainties in the system, instead of, as politicians use to think, and preach during the election campaigns, making it more predictable.
The above study of the market with restrictions on prices, is related to the problem which lately aroused considerable interest in theoretical economics. In 1970, G. Akerloff published an article that soon enjoyed much popularity. The article dealt with the markets with “asymmetric” information, i.e., markets, where the seller and buyer have different opportunities to estimate the quality of the item on sale.

Akerloff showed, further developing his contention in later publications, that, for asymmetric markets, the presence of low quality goods and dishonest sellers can result not only in sweeping away of high quality goods from the market, but may also result in a collapse of the market as such.

Briefly, Akerloff’s idea was as follows. Assume that the article of goods (say, a used car) may appear to be of low quality with certain probability, \( q \) (and, accordingly, of high quality with probability \( 1-q \)). Assume further that the buyer “knows” to an extent these a priori probability because \( q \) may reflect the ordinary index of production rejected by the factory.

But the seller in question knows about the article he or she is selling far better.

The buyer just follows the general opinions concerning the item. By this reason, the seller of the good second-hand car is seldom able to get the real price for it: the buyer wants to insure himself or herself against the dishonest seller.

The seller of the bad car, contrariwise, enjoys all the chances to get for it more money than it is really worth. As a consequence, good cars are “washed away” from the market, and the bad ones dominate.

Using the standard technique of the utility functions, Akerloff has shown that under certain conditions the equilibrium may be unattainable, e.g., when the graph of the supply's dependence on prices does not overlap with the graph of the demand's dependence on prices.

Akerloff interprets the problem of asymmetric markets linking it to the Grasham law, according to which the “bad” money oust the “good” money out of circulation. Akerloff conjectures that such an approach gives an opportunity to estimate the losses of the market from crooked dealings.

Akerloff concludes: the price of dishonest behavior amounts not just in the losses of the buyer, but also in the undermining consequences for honest businesses.

The model with restrictions on the prices from below discussed above, shed, in our view, some new light on the problem of asymmetric markets. We believe that Akerloff has proved not so much the possibility of collapse of the asymmetric market, but impossibility to apply the standard equilibrium techniques used in neo-classical analysis. Indeed, even in an asymmetric case, the flow of deals will last anyway, but for the study of the market state, the definition of equilibrium, based on the notions of statistical thermodynamics, looks to be more appropriate.

Indeed, consider two markets in interaction, one, with restrictions on prices from below, and another one, without such restrictions (say, a market of used cars). Consider a model with an “asymmetrical information”, we see that a market with restrictions on price from below is a market of "good" cars, while the market without restrictions is a market with defective cars (Akerloff used a slang term: "lemons") in circulation. As we have shown above, at a certain value of parameters of the flow of goods and "temperature" of the market (namely, at low temperature), the marginal price in the market with restrictions (here: in the market of "good" cars) becomes negative. This means that this market is collapsing. But the actual collapse will only happen at a rather low temperature. At higher temperature the market of high quality cars is quite viable.

Thus, it is possible to make some amendments to Akerloff's statements on the market of "lemons", and on the opportunities to operate honest business in the developing countries, where "standards of honesty" are low.
The market of "good" cars will not rarely be on the brink of collapsing but will not actually and totally collapse. According to our thermodynamic approach, the crucial factor lies not in the standards of honesty, but in the "low temperature" of the market. If the temperature is not low any more, the "honest market" (the one with restrictions on prices from below) is quite capable to coexist with the market of "lemons".

The marginal price formally corresponds to the pressure and the only thing needed to render the market "alive" is that the "pressure" in the market of "good" cars were not below a certain minimal value. For

Next, consider in more detail the thermodynamic model of the market with prices bounded from below. For this purpose, we apply the technique of the large statistical sum. But, first of all, let us make some remarks.

If we forget about the sellers and buyers, and only consider the amount of contracts, $N$, then the temperature of the market, according to our previous consideration, can be defined as:

$$ T = \frac{E}{N}, \tag{6.20} $$

where $E/\_\_$ is the amount of money.

Introduce the concept of a "potential contract" which means the possibility of striking a deal, with the goods flow $V_i = lV_0$, and money flow $E_k = kE_0$, where respectively $V_0$ and $E_0$ are least values of flows. Then some further simplification would be expedient. Introducing discrete time labeled by moments $T_k$, we assume that the goods and money flows only occur at discrete time moments. Such a model of the market resembles the market of von Mises. It allows us to treat every deal as a single-time event.

Bearing in mind the statistical aspect of the model, it looks quite natural to consider the market as the set of simultaneously coexisting systems covering all admissible deals, and to carry out averaging over the ensemble instead of averaging over time.

The potential contract $C(E_k, V_i)$ with parameters $V_i, E_k$ may be “filled in” with a real content, but also may remain “empty”. The notion of potential contract corresponds to the notion of an orbital in quantum mechanics. We are able now to refer to the set of potential contracts,

$$ C = (E_k, V_i) \text{ subject to some restriction, the contracts with } \frac{E_k}{V_i} < Q \text{ may be ruled out (which, actually, mirrors the restriction on prices from below, } Q \text{ being the least allowed price).}$$

Now, apply the standard technique of the large statistical sum. If $N << nm$, where $n, m$ stand for the total amount of the goods and money in the market, respectively, the market discussed is a “classical one”: the "population density" of every contract is very low, $\frac{N}{nm}$, so, we can neglect the probability that one “potential contract” may be “filled in” with two real contracts with equal $V_i$ and $E_k$. This means that, in the large statistical sum, we can neglect the terms of degree $\geq 2$ in $\{}$. Hence, the probability for the potential contract to be “filled in” is given by the formula:

$$ W(k, l) = \frac{k^n}{l^n T}. \tag{6.21} $$

If the number of actual contracts is equal to $N$, it is possible to find $\{}$ because $N$ can be found by summing over all probability values $W(k, l)$ for all potential contracts:

$$ \frac{N}{nm} = \sum \frac{W(k, l)}{k}. \tag{6.22} $$

Here $C(k)$ is the number of contracts with $E_k$, so:
\[ \mathcal{Q} = \sum_{k} \frac{N}{W(k,i)} . \]  

(6.23)

Since \( \mathcal{Q} = e^{\mathcal{Q}} \), where \( \mathcal{Q} \) is the migration potential, we can express \( \mathcal{Q} \) as follows:

\[ \frac{\mathcal{Q}}{T} = \ln \left( \sum_{k} \frac{N}{W(k,i)} \right). \]  

(6.24)

So, the entropy of the system can be derived from the formula:

\[ \frac{\partial S}{\partial N} \bigg|_{\mathcal{Q},V} = \frac{\mathcal{Q}}{T}. \]  

(6.25)

Substituting this value for \( \frac{\mathcal{Q}}{T} \), we find that:

\[ S = \sum_{0}^{N} \mathcal{Q} dN = \sum_{0}^{N} \ln N dN + N \ln \sum_{k} W(k,i), \]  

(6.26)

or

\[ S = N \ln N + N + N \ln \sum_{k} W(k,i). \]  

(6.27)

Now, differentiating the entropy with respect to \( V \) we derive the equation of state:

\[ \frac{P}{T} = \frac{\partial S}{\partial V} \bigg|_{E,N}. \]

It requires to calculate \( \mathcal{Q}(T,V) \).

First, consider the already discussed case, the one without restrictions on prices:

\[ \mathcal{Q}(T,V) = \sum_{k=1}^{n} C(k) e^{\frac{kE_{0}}{mT}}. \]  

(6.28)

Since a certain definite number \( m = \frac{V}{W} \) of potential contracts which can be “filled” exists for each value of \( k \), \( C(k) = m \). In order to simplify the calculations, we assume that \( n \) is very large \( (\mathcal{Q} = ne) \), and the upper limit of summation is equal to \( n \). We thus obtain:

\[ \mathcal{Q}(V,T) = \frac{V_{0}}{V} \frac{1}{e^{\frac{kE_{0}}{m}}} = \frac{V_{0}}{V} \frac{1}{e^{\frac{kE_{0}}{m}}} \]  

(6.29)

Hence, the already known result:

\[ \frac{P}{T} = \frac{\partial S}{\partial V} \bigg|_{E,N} = N \frac{\partial}{\partial V} \ln \mathcal{Q}(V,T) = \frac{N}{V}. \]  

(6.30)

Now, let us calculate \( \mathcal{Q}(T,V) \) for the case of prices bounded from below. The number of potential contracts \( C_{k}(E_{k},Q) \) now depends on \( k \) and the minimal price, \( Q \):

\[ C_{k}(E_{k},Q) = \sum_{q=1}^{kE_{0}/QW} 1 = \frac{kE_{0}}{QW}. \]  

(6.31)

In this case \( \mathcal{Q}(T,V) \) splits into two components.

One of them is obtained by summation up to \( k = \frac{mQW}{E_{0}} \), hence, \( C_{k}(E_{k},Q) = \frac{kE_{0}}{QW} \). For these values of \( k \), the number of potential contracts depends on \( k \). For very large \( k \) such dependence
vanishes since we assume that the amount of money used in any potential contract suffices to ensure any of \( m = \frac{V}{W} \) possible values of goods purchased:

\[
\varpi(T,V) = \prod_{k=1}^{QV} \frac{E_0}{E_k} \sum_{k = 0}^{QV} m^k e^{kE_0 / T} = \prod_{k=1}^{QV} \frac{QW}{W} e^{kE_0 / T} + \prod_{k=0}^{QV} m^k e^{kE_0 / T}.
\]

(6.32)

After simplifications we have:

\[
\varpi(T,V) = \varpi_1(T) \prod_{k=1}^{QV} \frac{E_0}{QW} e^{kE_0 / T} = \prod_{k=1}^{QV} \frac{E_0}{QW} e^{kE_0 / T}.
\]

(6.33)

where

\[
\varpi_1(T) = \prod_{k=1}^{QV} e^{kE_0 / T} = \prod_{k=1}^{QV} e^{E_0 / T} \frac{1}{e^{E_0 / T}}.
\]

Now, substitute \( \varpi(T,V) \) in the expression for the entropy, so the equation of state takes the form:

\[
\frac{P}{T} = \frac{\partial S}{\partial V} \bigg|_{\varpi} = \frac{\partial}{\partial V} \left( R(T,V) \right)
\]

(6.34)

or

\[
\frac{P}{T} = \frac{NQ}{E_0} \frac{1}{e^{QV / E_0} - 1}.
\]

(6.35)

The equation of state (6.35) is very interesting. When \( Q \ll \frac{T}{V} \), equation (6.35) turns into the familiar equation of the ideal gas:

\[
\frac{P}{T} = \frac{1}{V}.
\]

(6.36)

But, as it turns out, even small restriction on price from below completely transforms the equation of state.

In the range of larger \( V \), the marginal price \( P \) (i.e., the equilibrium price) becomes smaller than the minimal price of the contract, which means that the market collapses. Such a situation is equivalent to the negative marginal price for the free market. It is quite clear that if the least price is implemented, the large volumes of goods cannot be sold in equilibrium provided the temperature is bounded. This does not mean that the absence of deals. This means that the equilibrium cannot be achieved.

If \( T = \frac{E}{N} \), the minimal price is equal to \( Q \), and \( V = \frac{TN}{Q} = \frac{TN}{Q} \), the a part of goods equivalent to \( V \left\lfloor \frac{T}{Q} \right\rfloor \) is impossible to realize.
A hasty observation may lead to the premature conclusion that the critical mass of the goods when the price is bounded from below will lie in a neighborhood of:

$$V_{\text{crit}} = \frac{T N}{Q}.$$  \hspace{1cm} (6.37)

But, as is clear from the above analysis, this is not the case: in reality the critical mass is considerably smaller. Namely, assuming that

$$P = \frac{T N Q}{E_0} < \frac{1}{e^{1/T}} < Q$$  \hspace{1cm} (6.38)

or

$$\frac{T N}{E_0} < e^{\frac{Q_{\text{crit}}}{T}} \frac{1}{1},$$  \hspace{1cm} (6.39)

we obtain

$$V_{\text{crit}} = \frac{E_0}{Q} \ln \left( \frac{T N}{E_0} \right) \frac{1}{1}.$$  \hspace{1cm} (6.40)

As a result, instead of $V_{\text{crit}} = \frac{T N}{Q} = \frac{n E_0}{Q}$, deduced by the “naive” argument, $V_{\text{crit}}$ is close to $\ln V$, i.e., is of several orders of magnitude smaller. Therefore, the market collapses much earlier than it seems at the first glance.

Coming back to the market with “lemons”, this means that the ousting of good cars by the bad ones will happen much earlier than the naive a priori assumptions allow for.

Clearly, the asymmetrical information narrows the market’s “zone of stability” to a very small size. Thus, our analysis justifies Hayek’s conjecture that the market exists thanks to the culture of “honest business”, and survives in the process of competition of business cultures.

The investigation above also points to another condition for existence of markets with high quality goods. The market operating with high quality goods has to be somehow separated from the market of low quality goods. Otherwise, with the influx of high quality commodities, the market becomes unstable.

The model outlined above shows that due to a limited supply of high quality goods, their market can coexist and interact with the market of “bad” goods, but only as the “market for elite”, which means that outside its edges the market is stunted by the flaws in the “business code of behavior”. An alternative is certification of goods.

We see that in a sense, our theory excludes "human component": the equilibrium theory in economics becomes parallel to that of statistical physics.

The approach we suggest is not supposed to be applied to technologies _an object of human activity._

It is also clear that hypothesis on equal probability of market states _the basis of any statistical theory _may be justified only if we, following Hayek, consider the human activity a principal factor subject to incomplete information on what's on at the market. Otherwise the hypothesis on equal probability of market state becomes absurd.

Human components is essential in one more aspect. The market requires abiding certain rules of contracts. In our theory we assume this as a given reality. This, however, is extremely important. Even if contracts are not restricted by "external" rules, the mere necessity to abide the contracts makes the market into a social institute. The behavior of this institute, considered as a machine, in particular, its thermodynamic properties are of huge interest, but first such a machine _such an
institute _ must be created and being created should be maintained. Its collapse is a universal disaster.

§7 Two markets with two items of goods

Consider now the problem of the market equilibrium for the two markets with two items of goods, capable to replace each other. The market with replaceable goods poses one of the most known problems of the mathematical economics. The study of such markets prompted the marginalist revolution.

In terms of the neo-classical theory, to work out this task, it is necessary to define the utility functions of the goods, the function depending on the volume of consignments. When the derivative of the utility function with respect to the volume of consignment decreases as this volume grows, the techniques of the classical analysis show that the summary utility function reaches its maximum at such a volume of the consignment of goods that the increase of utility by one unit of the expenditure is equal for all nomenclature of goods.

Intuitively, this is a plausible statement. If, at the expenditure of one unit of funds per one item of goods, it were possible to increase the total utility of the purchased goods by replacing one item with another, this replacement have been implemented, until the utility has not dropped due to the increase of the good's volume.

We see that the dependence of utility on volume is indeed very important in terms of maintenance of sustaining the system’s stability: otherwise all the funds could have been invested in the most useful goods only.

Let us now study the problem of replaceable goods in the framework of thermodynamic approach.

Clearly, it makes sense to speak about replacement only if there is a certain equivalence between the goods. If the is no equivalence, we cannot proceed. So, let such an equivalence exist in the model, i.e., \( n \) units of item 1 are equivalent (never mind, in what sense precisely) to \( m \) units of item 2. The equivalence relation can be used to find the maximum of entropy, in exactly the same manner, as it was done before for only one item of goods.

It is important to observe that, in our hypothesis, the equivalence relation does not depend on the flow of goods. If such dependence takes place, of the theory of statistical equilibrium is, all the same, possible to deduce, but it is a bit more involved.

The most probable or, what is the same here, equilibrium state of the system is achieved, as the entropy of the system attains its maximum. To find the maximum of entropy, we use the equivalence relation between two items of goods, in completely the same fashion as earlier on these pages.

Thus, consider a market with \( N \) agents, with a flow of money \( E \) and flows of items of two goods, \( V_A \) and \( V_B \). To find the entropy of system, we have to count the number of probable deals, i.e., the number of ways to attribute to each agent four numbers \( (E_A, E_B, V_A, V_B) \) that show how much money was spent by an agent per unit time on purchasing the goods \( A \) and \( B \), and how many goods had been bought over the same period, respectively. Adding up over all agents we obtain the values of macroscopic parameters of the system.

Observe that in order to estimate the system’s state, it does not suffice to simply fix the amount of money spent by the \( i \)th agent because some microscopic states may be ruled out, owing to, say, a restriction of the prices from below.
In the general case, the dependence of the entropy on macroscopic parameters cannot be represented as the sum of components each depending on one or two parameters. For the free market, however, this is so.

Consider the system with entropy \( S(E_A, E_B, V_A, V_B) \). Replacing \( E_A \) with \( E_B \) and making use of the fact that \( E_A + E_B = E \) we see that

\[
\mathcal{D}S = \frac{\partial S}{\partial V_A} V_A + \frac{\partial S}{\partial V_B} V_B = \frac{\partial S}{\partial A} A + \frac{\partial S}{\partial B} B.
\]  

(7.1)

In the state of equilibrium, the derivatives \( \frac{\partial S}{\partial V_A} \) and \( \frac{\partial S}{\partial V_B} \) are equal and \( V_A \) and \( V_B \) are fixed. If there is an equivalence between the items \( _A \) and \( _B \), we may "unite" these items and consider the equilibrium problem. Let \( V_A = V_A^n n_A, V_B = V_B^0 n_B \) where \( n_A \) and \( n_B \) are the amounts of goods, purchased per unit time.

Having introduced \( V_A \) and \( V_B \) we imply that the replacement relation is known, i.e., we have a common unit of measurement. So, again, we have:

\[
\mathcal{D}S = \frac{\partial S}{\partial V_A} V_A + \frac{\partial S}{\partial V_B} V_B = \frac{\partial S}{\partial V_A} A + \frac{\partial S}{\partial V_B} B,
\]  

(7.2)

and the condition for the equilibrium state can be expressed as:

\[
\frac{\partial S}{\partial V_A} = \frac{\partial S}{\partial V_B}.
\]  

(7.3)

Observe that we made no assumptions on utility, but only on possibility to replace the goods with other goods.

Now consider the free market where we know how the entropy dependence on macro-parameters. As we outlined above, the equilibrium conditions with respect to the money flow are of the form:

\[
\frac{\partial S}{\partial E_A} = \frac{N}{E_A}, \quad \quad \frac{\partial S}{\partial E_B} = \frac{N}{E_B}.
\]  

(7.4)

As \( E_A + E_B = E \), the equality of \( \frac{\partial S}{\partial E_A} \) and \( \frac{\partial S}{\partial E_B} \) result in \( E_A = E_B = \frac{E}{2} \). Similar calculation techniques for \( \frac{\partial S}{\partial V_A} \) and \( \frac{\partial S}{\partial V_B} \) produce \( \frac{\partial S}{\partial V_A} = \frac{N}{V_A} \) and \( \frac{\partial S}{\partial V_B} = \frac{N}{V_B} \) and, with the constraint that \( V_A + V_B = V, V_A = V_B = \frac{V}{2} \). Returning to the condition of substitution, expressed by \( V_A = V_A^n n_A, V_B = V_B^0 n_B \), it means:

\[
\frac{n_A}{n_B} = \frac{V_A^0}{V_A^n},
\]  

(7.5)

i.e., the volume of goods, purchased in the equilibrium state of the free market, is inverse proportional to its "replacement capacity".

Recall our earlier assumption that the equilibrium of the money flow is achieved. The conditions for the equilibrium on the goods flow takes the form:

\[
T \frac{\partial S}{\partial V_A} \bigg|_{E_A, E_B} = T \frac{\partial S}{\partial V_B} \bigg|_{E_A, E_B}.
\]  

(7.6)
If the money flow is constant, $TdS$ can be interpreted as the expenditure on the purchase of the last, “marginal” portion of the goods. So, we have obtained the well-known marginalists's formulation:

In the state of equilibrium, the amount of money spent on the last portion of goods per unit of replacement capacity is the same for all items of goods represented at the market.
Bibliography

1 It is worth remembering, among other things, that the subject of the modern research in mathematical economics is, as a rule, the state of equilibrium in itself. The dynamics of the system is only considered as a metaphor, pointing out the way this equilibrium state may be achieved. See Fisher F.V. Disequilibrium Foundations of Equilibrium Economics. Cambridge: Cambridge Univ. Press, 1963, p.3.

See also such classical works as: Wald A. On Some Systems of Equations of Mathematical Economics. Econometrica, 1951, 19, p.368-403;

In general, further evolution of the theory produced such a state of affairs, in which the system’s dynamics in the vicinity of the equilibrium point was completely neglected by the researchers.


ix In the end of 1960s _ Wilson suggested that the entropy analysis should be used for the examination of transport flows. (Wilson A.G.. Entropy in Urban and Regional Modeling. London: Pion, 1970.). Wilson mainly used the method of the maximization of entropy to tackle transportation problems. The use of entropy approach is also well-known in the economic studies in relation to the estimation of uncomplete data on the basis of the ideas by E.T. Jaynes:

For details see

These works do not include, however, analysis of economic equilibrium. On the precedents of application of other methods from statistical physics for socio-economic studies, see, for instance,


This idea was suggested by Frenkel J., see J. Frenkel *Principles of the Theory of Atomic Nuclei*. Moscow, 1955 (in Russian)


The problem of unemployment is of utmost importance for the discourse on economic balance, as it represents by itself a “counter-example” for neo-classical theory of balance, stating that no excess demand is possible. This served as one of greatest incentives for the emergence of institutional approach to the economy. See:


