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Abstract. The relation between stable (zero-gradient) points and cellular automata dynamics is explored. Both the number and the nature of stable points influence to some extent the complexity class and stability of rules, and the existence of spatio-temporal structures such as gliders.

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Recently cellular automata (CA) have played two important roles. As totally discrete models (in space, time and state variable) they have proved very useful for the computer simulation of diverse spatially extended problems in physics, chemistry and biology [1-5]. They have also been an important tool in the classification of complex spatio-temporal behavior [6] and in the understanding of the origins of complexity in physical systems [7].

This Letter addresses the related problem of understanding CA dynamics in terms of properties of CA rule tables, to be described below. We concentrate on the four Wolfram Classes of behavior [6]: Class I consists of rules with a homogeneous asymptotic state, Class II has stable spatial domains, gliders or very short period patterns, Class III displays chaotic patterns, and Class IV has long-lived spatial structures that interact in complex ways. We also consider specifically spatio-temporal structures such as domains, dislocations and gliders.

Rule tables express CA dynamics in terms of the (simultaneous) updating of site values, chosen from a discrete set, given as a function of the configuration near each site (neighborhood). Here we consider one-dimensional deterministic CA with radius one (three-site neighborhoods) and two states per site (0,1), known as *elementary* CA. A subset of radius-two rules will also be studied. The table for elementary CA rules contains $2^3 = 8$ entries. We use the Wolfram notation for rules – the decimal equivalent of the eight neighborhood outputs. For example, if the central site evolves to one only if exactly one of its two neighbors is one, $f(001, 011, 100, 110) = 1$ and $f(000, 010, 101, 111) = 0$. Ordering the neighborhood outputs from (111) (highest bit) to (000) (lowest bit) yields the rule table (01011010), or 90 in decimal notation. Because of left-right and 0-1 symmetries, only 88 out of the $2^8 = 256$ elementary rules are dynamically independent. Hence, later statements about neighborhood (000) also apply to (111). Two other sets of neighborhoods are also dynamically equivalent: (101,010) and (001,100,011,110).

Some progress has been made in the understanding of CA dynamics based on the rule table. For example, Langton and coworkers [8] identified the fraction of non-zero outputs in the rule table as a reasonable predictor of the single-site entropy of a rule for a sufficient number of states and neighbors, if the strong quiescent property $f(00\ldots00) = 0,\ f(11\ldots11) = 1$ is satisfied. Li and Packard [9] made the important observation that some neighborhoods, such as $(00\ldots00)$ and $(11\ldots11)$, control the dynamics more than others. These are known as hot bits.

More recently *derivatives* of the rule table have also been shown to provide important complementary information about CA rules. The notion of Boolean derivatives for CA was introduced by Vichniac [10]. The Boolean derivative of f with respect to bit x_j is $\delta f/\delta x_j = (x_1, ..x_j, ..) \otimes f(x_1, ..\neg x_j, ..)$, where $\otimes$ represents the exclusive or operator and $\neg$ represents the negation operator. Simply put, the Boolean derivative is 1 if f is sensitive to a change in the bit x_j, and 0 otherwise. A Lyapounov exponent for CA has been proposed recently [11]. Also, a global sensitivity parameter, which averages the Boolean derivative over all neighborhoods and sites [12] has been used in constructing diagrams of rule behavior. In the same vein, Kohring [13] proposed that the number of stable points, those for which $\nabla f = 0$, can be used to classify CA, as they should dictate to some extent CA dynamics. Information about stable points is independent from and complementary to that provided by an average sensitivity parameter [12]. In this respect, stable points play a role analogous to hot bits.

After examining carefully a number of one-dimensional CA, we find that (1) there is indeed a good statistical correlation between Wolfram Classes and the number of stable points, but a wide range within each Class, (2) stable points with the property that the output differs from the central bit, henceforth called anti-stable points, tend to produce chaotic rules or rules with periodic patterns, (3) to have Class I behavior it is necessary but not sufficient that (000) be a true stable point (i.e., not anti-stable), and (4) certain stable points tend to control the existence of domains or gliders. Further, statements (1) and (3) are confirmed by examining a particular subset of radius-two rules.

All the (anti-)stable points associated with each and all independent elementary CA rules were identified. Three rules (73, 94 and 108) are difficult to classify. Three others (54, 62 and 110) show some symptoms of Class IV behavior, and have no stable points. After eliminating these rules, 8 Class I rules, 62 Class II rules and 12 Class III rules remained.

Number of stable points vs. Class. Out of the easily classifiable rules the number of stable points for Class I is 3.37 ± 1.0, with range between 1 and 8. For Class II the number is 1.15 ± 0.98 with range between 0 and 4. For Class III the number is 0.42 ± 0.90, with range between 0 and 3. One can also assign a value of 1 to stable points, and -1

to anti-stable points. In this case, the number of points for Class I is 0.625 ± 1.41, with range between -2 and 2. For Class II it is -0.097 ± 1.35 with range between -2 and 3, and for Class III it is -0.4167 ± 0.9 with range between 0 and -3. Therefore for this family of CA rules dynamics and the number of stable points are correlated statistically, but not rule-by-rule. We note that Class III rules seem to require the absence of true stable points. Rules with no true stable points are found in all Classes except Class I. In fact, we emphasize that the information provided by stable points appears to only partially correlate with dynamical behavior. For example, domains are quite possible without stable points, just by virtue of the fortuitous collective effect of the rule table. Rule 28 has no stable points, and yet the fact that $f(010) = f(011) = f(100) = 1,\ 0$ otherwise, makes 01 strings act as unchangeable domain walls.

Stable point 000. Class I rules provide a good example of how not only the number but the *nature* of stable points is important: asymptotic homogeneous states are found with even just one stable point. Closer examination reveals that what is important is *which* neighborhood is stable. It turns out that, as long as (000) is a true stable point, rules are Class I, or in a few cases Class II with wide domains of ones. If sufficiently few neighborhoods iterate to 1, blocks of ones tend to erode. If enough do iterate to one, including (101) and (111), thick domains of ones form. Of the 88 independent rules only one (23) shows (000) as an anti-stable point; this rule iterates to domains of alternating ones and zeros.

Class III blocked patterns. Reference [3] shows the evolution of elementary CA rules with only every second site depicted. This helps to visualize certain patterns. In particular, some symptoms of Class IV behavior are shown by rules 18, 41 and 146, which are three of the four Class III rules which have anti-stable points. No explanations for this have been found.

Stable points and spatial structures. It has already been determined that elementary rules with stable points do not exhibit Class III behavior, and conversely, that a particular stable point is needed for Class I behavior. We concentrate now on Class II rules, which usually exhibit either domains, gliders, or dislocations (i.e., moving boundaries between domains). One finds that the average number of stable points (including anti-stable) is 1.14 ± 0.88 for gliders or dislocations, and 1.15 ± 1.12 for domains. In both cases the range is between 0 and 4. Thick domains have already been discussed. While no particular stable point appears to be associated with the existence of domains, there are seven rules (1, 19, 23, 50, 51 and 178) which exhibit period-two domains. Anti-stable points, especially (111) and (110), appear to be associated with these time-periodic domains. With gliders or dislocations, the anti-stable point (110) is present in almost half (15) of the thirty-three instances. No additional information emerged specifically about dislocations, or from giving anti-stable points the value -1.

The stability of Class II rules. An important signature of rules is the difference pattern, i.e. what happens if a configuration is modified by one or a few bits and the resulting evolution is compared with the undisturbed one. Usually differences die away in Class I rules, and spread at the maximum possible rate (related to neighborhood radius) for Class III rules. In Class II rules, the difference patterns can die, appear as domains or gliders, or in a few cases display a localized but tortuous pattern (25, 35, 43 and 142). To some extent it appears that difference patterns are regulated by the number of stable minus anti-stable points. It is found that rules with difference patterns which die out have 1.25 ± 1.03 stable points, those with domains 0.04 ± 1.58 stable points, those with gliders -0.53 ± 0.86 stable points, and those with tortuous patterns -0.5 ± 1.7 stable points. The difference between the average number of stable points in domain and glider difference patterns has not been explained. Generally, our findings indicate that the number of stable points is somewhat related to the *stability* of rules: more stable points result in simpler difference patterns.

Totalistic, quiescent, radius-two rules. Two-state, radius-two (e.g., neighborhood of size five) rules in which the output depends only of the sum of neighbors and the quiescent property $f(00000) = 0$ is satisfied, have been studied by Wolfram [6]. The rule notation is now a six-bit number, expressing the outcome of sum-six, five, ... , zero neighborhoods. The number of (anti-)stable points for these rules have also been counted in order to examine the generality of the observations made in elementary CA. The average number of stable, and stable minus anti-stable points, given as two numbers with the respective ranges in parentheses, are as follows: Class I: 16.6 (1 to 32), 7.4 (-5 to 32); Class 2: 5.2 (1 to 10), 4.0 (1 to 8); Class 3: 2.4 (0 to 10), -0.81 (-6 to 3); Class IV: 1.5 (1 to 2), 1.5 (1 to 2). Hence, the dynamics – number of stable points correlation is again validated in a statistical sense. Several differences arise with respect to elementary CA. The true Class IV rules exhibit few (1 or 2) stable points, as opposed to none, or a few anti-stable points for the near-Class IV elementary rules. We also find an instance of a Class III rule which has (00000) as a true stable point (rule 101100 in the notation of [6]), and several Class III rules with more anti-stable than true stable points (e.g., rule 000100 in the notation of [6]). Because of the quiescent property and the symmetries in these rules, statements about gliders and about (00000) as an anti-stable point could not be verified with these rules.

Discussion. CA dynamics and the number of stable points in their rule table have been shown to be correlated, but only in a statistical sense. The wide range for the number of stable points within any Class should remind us that this number only provides *partial* information which should be complemented by other properties of the rule table [8-12]. However, a few patterns have emerged. In particular, we have seen that, for the rules studied, Class I behavior only requires one stable Point: (000). This point can also cause wide domains of ones, but only if a sufficient number of other neighborhoods also iterate to one. Conversely, Class III behavior requires the absence of true stable points in elementary CA. This condition is not required in larger-radius rules. Anti-stable points seem to be associated with Class III, with period-two domains, and with gliders and dislocations; the latter are promoted by anti-stable point (110). In addition the stability of Class II rules, specifically the form of difference patterns, also appears to be related to the difference between the number of stable and anti-stable points. Class IV rules seem to require very few or no stable or anti-stable points. It is unclear to what extent these observations will hold for rules with larger neighborhoods, more states per site, or in higher dimensions. In particular, it has been established by random sampling of rules that for larger neighborhoods Class III rules become more prevalent and Class II rules are less common [6]. The conclusion is that Kohring's proposal to classify CA rules by their number of stable points is useful, but complementary criteria are needed.

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