Community Detection with the z-Laplacian

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Abstract

Community detection is a fundamental problem in network science, with broad applications across the biological and social arenas. A common approach is to leverage the spectral properties of an operator related to the network (most commonly the adjacency matrix or graph Laplacian), though there are regimes where these techniques are known to fail on sparse networks despite the existence of theoretically detectable community structure [3],[4]. This work introduces an operator we term the "z-Laplacian" $L_z = zA - D$, which has been observed to share important spectral properties with the non-backtracking matrix of [3],[6] and which we believe can find communities even in the sparse case. We augment tools from the theory of random matrices with message-passing and population dynamics approaches in order to study the spectrum of L_z.

1 Spectral Clustering

A celebrated method for finding network community structure is to classify vertices according to the eigenvectors of a linear operator associated with the graph such as the adjacency matrix or Laplacian. However, it has been obesrved that these spectral methods are in general ineffective when the network is very sparse [3],[4]. Krzakala et al. propose a spectral algorithm in [3] based on the non-backtracking matrix B, which represents a walk on the network where backtracking along an edge is prohibited; they argue that this method successfully extracts community structure even when the network is sparse. In this work, we explore the spectral properties of yet another linear operator—the z-Laplacian—which shares spectral properties with B and which we believe is also suitable for detecting communities on sparse networks. A similar operator was used in [6] to study the spectral properties of B.

Random Graphs, Random Matrices

It is common practice to study the behavior of spectral methods when applied to various random graph ensembles; we focus on two in particular. In the **Erdős-Rényi** model, denoted $G(n, p)$, we choose the number of vertices n as well as the average degree c, and include each of the possible edges with probability $p = \frac{c}{n}$ [2]. We can think of this process as inducing a probability distribution on the space of all possible graphs on n vertices. Often we are concerned with the behavior of these graphs in the limit as n tends to infinity, and distinguish the **dense** case, where $c = O(n)$, from the **sparse** where $c = O(1)$.

The adjacency matrix A of a $G(n, p)$ graph is a random matrix where each diagonal entry is zero and each off-diagonal entry is Bernoulli-distributed with parameter p. This characterization allows us to bring to bear the tools of random matrix theory in our study of the spectra of $G(n, p)$ adjacency matrices. A foundational result concerns the **empirical spectral density** A, which we write

$$\rho(\zeta) = \frac{1}{n} \sum_{j=1}^{n} \delta(\zeta - \lambda_j),$$

summing over the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of A; we can apply a theorem of Wigner to show that $\rho(\zeta)$ for the ensemble $G(n,p)$ converges in expectation to the well-known Semicircle Law

$$\rho_{SC}(\zeta) = \frac{1}{2\pi c}\sqrt{4c - \zeta^2} \tag{1}$$

in the large-n limit [7].

Although any graph on n vetices is a possible outcome of the Erdős-Rényi model, graphs with community structure occur with vanishingly small probability. A random graph model for which the typical graph has community structure is the **Stochastic Block Model,** where we first fix the number of nodes n, number of groups k, and a vector of a priori group assignments $\boldsymbol{q} \in \{1, ...k\}^n$, and include each edge with probability dependent on the group assignments of the two nodes it joins. We encode these probabilities in a matrix $P \in [0,1]^{k\times k}$, where P_{ij} gives the probability of an edge joining a group u and group j vertex. Again we are interested in the large-n limit, and distinguish the sparse from dense cases as follows. Let $P = \frac{1}{n}C$, where C_{ij} gives the expected *number* of neighbors in group j that a group i vertex will have; taking $C_{ij} = O(n)$ results in dense graphs and $C_{ij} = O(1)$ sparse.

Again we can think of the adjacency matrix of an SBM graph as a random matrix with Bernoulli-distributed variables, though in this case the group assignments define a block structure and the Bernoulli parameter in each block is determined by the matrix P. However, the generality of the Wigner theorem allows us to once again prove that the empirical spectral density converges to (1) [4].

The Detection Threshold

Because of the randomness inherent in the SBM, there is a parameter regime in which edges are empirically more likely to occur between vetices of the same group, but these groups are undetectable to *any* algorithm. We follow [3],[4] in considering the case where there are only two groups of equal size,

$$C_{ij} = \begin{cases} c_{in} & i = j \\ c_{out} & i \neq j \end{cases},$$

and we call c the average degree of the network; it has been shown that unless

$$c_{in} - c_{out} > 2\sqrt{c}, \tag{2}$$

the community structure is impossible to detect.

For graphs that are sufficiently dense, a closer look at the spectrum of the adjacency matrix gives some insight into this threshold. Asymptotic convergence to (1) means that most of the spectrum—the 'bulk'—is confined to the interval $[-2\sqrt{c}, 2\sqrt{c}]$ even for finite n. However, the largest eigenvalue λ_1 always lies above $2\sqrt{c}$; its associated eigenvector encodes vertex centrality in the graph. If the graph is sufficiently dense, there is a second eigenvalue λ_c which is separated from the bulk; its associated eigenvector sorts vertices by community membership. This eigenvalue is

$$\lambda_c = \frac{c_{in} - c_{out}}{2} + \frac{c_{in} + c_{out}}{c_{in} - c_{out}};$$

we recover (2) by looking for the condition on c_{in} and c_{out} that separates λ_c from the bulk [3],[4].

Sadly, we are not so fortunate in the sparse case, where, as Figure 1 shows, the spectrum deviates significantly from the Wigner Semicircle and upper edge of the bulk overtakes λ_c, making it impossible to locate among the bevy of meaningless eigenvectors even when (2) is satisfied. Krzakala et al. demonstrate in

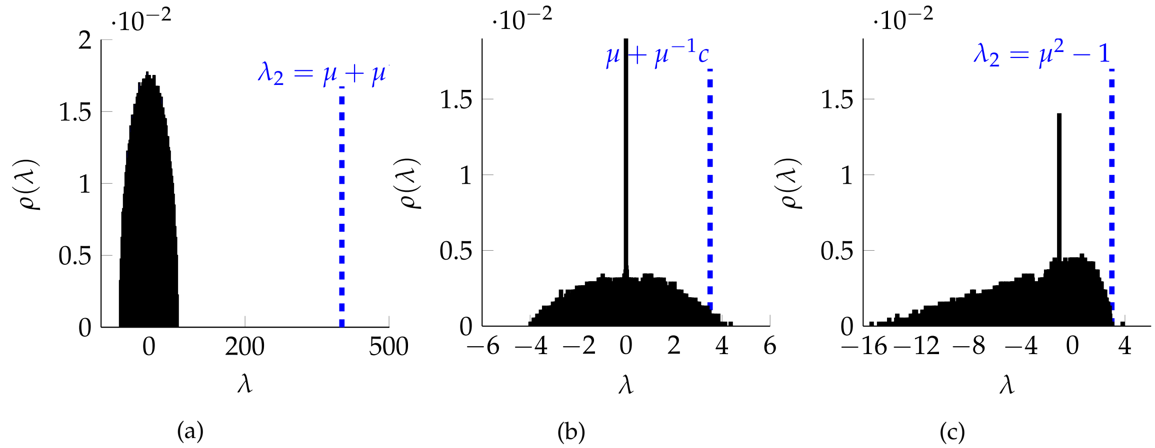


Figure 1: *(a) Spectrum of a dense SBM graph, $n = 4000$, $c_{in} =$, $c_{out} =$. Most eigenvalues are confined to the interval $[-2\sqrt{c}, 2\sqrt{c}]$, and the community eigenvalue is well-separated from the bulk. (b) Spectrum of a sparse SBM graph, $n = 4000$, $c_{in} =$, $c_{out} =$. The bulk has overrun the community eigenvalue despite theoretically detectible community structure. (c) Spectrum of L_z for the same sparse SBM graph. The bulk has been shifted to the left, leaving the community eigenvalue $\mu^2 - 1$ exposed.*

[3] that spectral methods using the 'non-backtracking perator' B succeed in detecting community structure whenever (2) holds; the z-Laplacian operator that we propose in the sequel is inspired by [3] and we conjecture that it as well succeeds down to the detection threshold. Figure 1 shows that it does so by skewing the bulk toward $-\infty$ and exposing a community-correlated eigenvalue.

The z-Laplacian

We motivate the z-Laplacian as follows. It has been noted ([3],[6],[5]) that that *every* eigenvalue λ of the nonbacktracking matrix B differet from ± 1 satisfies

$$\det \left[\lambda^2 \mathbb{1} - \lambda A + (D - \mathbb{1}) \right] = 0$$

and thus we define the z-Laplacian to be, for any z,

$$L_z = zA - D. \tag{3}$$

Krzakala et al. show that a particular vector $\boldsymbol{f}$ correlated with the community structure approximately satisfies the quadratic eigenvalue equation

$$A\boldsymbol{f} = \mu \boldsymbol{f} + \mu^{-1}(D - 1)\boldsymbol{f},$$

where $\mu = \frac{c_{in} - c_{out}}{2}$ and D is the diagonal degree matrix [3]. Applying some algebra, we see that this vector is thus an eigenvalue for L_z with eigenvalue $\mu^1 - 1$; Figure 1 shows that this eigenvalue is not buried in the bulk for L_z. Our task is then to understand the spectrum of L_z, and specifically to find a regime of z where $\mu^2 - 1$ is well-separated from the upper edge of the bulk.

2 The Spectral Density as a Generating Function

In order to study the spectrum of L_z, we will for the moment focus on graphs generated by $G(n, c/n)$ where $c = O(1)$. We first record a fundamental result from the theory of random matrices relating the empirical spectral density of a matrix $\boldsymbol{M}$ to the traces of its powers. Recall that the empirical spectral density of $\boldsymbol{M}$ is

$$\rho(\zeta) = \frac{1}{n}\sum_{j=1}^{n} \delta(\zeta - \lambda_j).$$

Expressing $\delta(\zeta)$ as a limit of Cauchy distributions and exploiting some complex arithmetic, we see that

$$\delta(\zeta) = \lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{\zeta^2+\epsilon^2} = \lim_{\epsilon\to 0} \frac{1}{\pi}\Im\frac{1}{\zeta - i\epsilon},$$

meaning that we can rewrite the spectral density as

$$\begin{aligned}
\rho(\zeta) &= \frac{1}{n\pi}\lim_{\epsilon\to 0}\Im\sum_{j=1}^{n}\frac{1}{\zeta-\lambda_j-i\epsilon} \\
&= \frac{1}{n\pi}\lim_{\epsilon\to 0}\Im\frac{1}{\zeta}\sum_{j=1}^{n}\sum_{t=0}^{\infty}\left(\frac{\lambda_j+i\epsilon}{\zeta}\right)^t \\
&= \frac{1}{n\pi}\lim_{\epsilon\to 0}\Im\sum_{t=0}^{\infty}\frac{\operatorname{tr}(\boldsymbol{M}+i\epsilon\mathbb{1})^t}{\zeta^{t+1}}.
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ from above, we can finally write

$$\rho(\zeta) = \frac{1}{n\pi}\Im\sum_{t=0}^{\infty} g(\zeta), \tag{4}$$

where

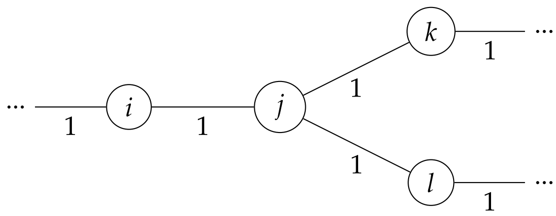
$$g(\zeta) = \sum_{t=0}^{\infty}\frac{\operatorname{tr}\boldsymbol{M}^t}{\zeta^{t+1}}. \tag{5}$$

Interpreting $\boldsymbol{M}$ as a (weighted) adjacency matrix,and recalling the fact that $\operatorname{tr}\boldsymbol{M}^t$ counts the number of (weighted) loops of length t on the graph which $\boldsymbol{M}$ encodes, we can think of $g(\zeta)$ as the generating function which counts loops.

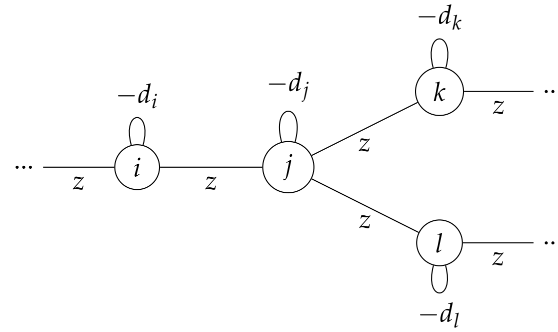
3 Message-Passing

With equations (4) and (5) in hand, we can study the spectrum of L_z by interpreting L_z as a new graph $G' = (V', E')$ and computing the generating function $g(\zeta)$ for loops on G'. Referring to the definition $L_z = zA - D$, it's clear that we can obtain G' from G by adding a self-edge at every vertex v_i with weight d_i and giving each of the original edges a weight z. Moreover, we can leverage the fact that sparse Erdős-Rényi and SBM graphs converge to trees [cite] in the large-n limit in order to compute $g(\zeta)$ via a message-passing approach.

Let $g^{i\leftarrow j}(\zeta)$ be the generating function for loops starting and ending at v_i and not using the edge (i, j). Assuming that the graph is locally treelike, we can think of such loops as being made up of two components. The first are self-edges at v_i with weight $-d_i$, the second—for each neighbor v_k of v_j other than v_i—are loops $(v_j, v_k, ..., v_k, v_j)$ such that the the edge (v_k, v_j) does not appear in the loop $v_k \to v_k$; these have weight $z^2 g^{j\leftarrow k}(\zeta)$. Every loop counted by $g^{i\leftarrow j}$ is a concatenation of these two component loops of some arbitrary



(a)



(b)

Figure 2: The original graph G (a) and graph G' represented by L_z (b)

length, so we can write $g^{i\leftarrow j}$ as

$$
\begin{aligned}
g^{i\leftarrow j}(\zeta) &= \sum_{r=0}^{\infty} \frac{1}{\zeta^{r+1}} \left(-d_j + z^2 \sum_{k\in\partial(j)\setminus i} g^{j\leftarrow k} \right)^r \\
&= \left(\zeta + d_j - z^2 \sum_{k\in\partial(j)\setminus i} g^{j\leftarrow k}(\zeta) \right)^{-1} .
\end{aligned} \tag{6}
$$

Using this recursive equation, we can compute $g^{i\leftarrow j}$ for every pair of vertices v_i and v_j by treating $g^{i\leftarrow j}$ as a message sent from v_j to v_i; we initialize the population of messages with random complex numbers, update the messages that each vertex sends according to the messages it receives from its neighbors, and iterate this process until the message population converges. At this point we can compute the generating function for *all* loops starting and ending at vertex v_j by

$$
\begin{aligned}
g^{j}(\zeta) &= \sum_{r=0}^{\infty} \frac{1}{\zeta^{r+1}} \left(-d_j + z^2 \sum_{k\in\partial(j)} g^{j\leftarrow k} \right)^r \\
&= \left(\zeta + d_j - z^2 \sum_{k\in\partial(j)} g^{j\leftarrow k} \right)^{-1} ,
\end{aligned}
$$

making

$$
\rho(\zeta) = \frac{1}{n\pi}\Im g(\zeta) = \frac{1}{n\pi}\Im \sum_{j=1}^{n} g^{j}(\zeta). \tag{7}
$$

However, (7) means that we can locate regions where L_z has zero spectral density without computing $g(\zeta)$ at all, simply by looking for values of ζ at which all of the messages $g^{i\leftarrow j}(\zeta)$ are real. Of particular salience to the community-detection properties of L_z is that the upper edge of the bulk occurs at the smallest value of ζ for which the message population is real. Even for $c = O(1)$, graphs generated by $G(n, c/n)$ are in general not trees for $n < \infty$; in this case the message-passing approach allows us to make an estimate of the edge of the bulk which in the infinite limit appears to be exact (Figure 3).

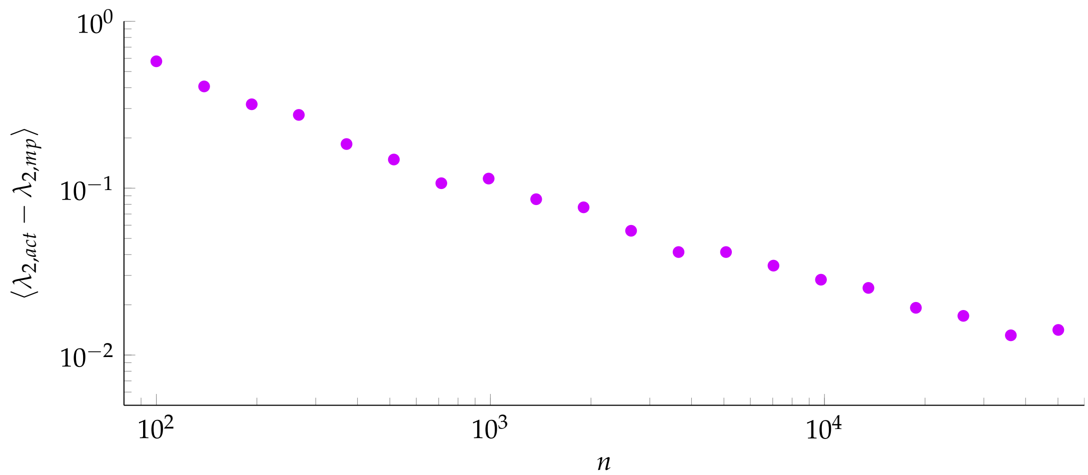


Figure 3: *Difference for L_z between the actual edge of the bulk (λ_2) and the numerical message-passing approximation given by* $\min_\zeta \left\{ \Im g^{j \leftarrow k}(\zeta) = 0 \right\}$*, for $G(n, c/n)$ with $c = 3$, $z = 2.366$, and n ranging from* 200 *to* 50,000*. Differences for each n are averaged over 20 graph realizations.*

4 Population Dynamics

In order to compute the average spectral density for an entire random graph *ensemble*—i.e. $G(n, p)$ for fixed n, p—we can alter this message-passing machinery as follows. We begin with an initial population of messages $\{g_j(\zeta)\}$ and at every step we pick D from the degree distribution of random graph ensemble, select $D - 1$ messages $g_{j_1}(\zeta), ..., g_{j_{D-1}}(\zeta)$ from the population, compute

$$\hat{g} = \left(\zeta + D - z^2 \sum_{k=1}^{D-1} g_{j_k} \right)^{-1}, \tag{8}$$

and use $\hat{g}$ to replace a random message from the population. This process is repeated until the *distribution* of messages reaches a fixed point. Although it is possible to compute the generating function $g(\zeta)$ from this limiting message distribution, once again we can locate regions of zero spectral density by finding values of ζ for which the limiting distribution is real-valued.

The δ-Function Message Distribution

While the distributional fixed point of (8) is in general quite complicated, there is a critical value of ζ at which the limiting message distribution is a δ-function where all the messages have the same value. If $g_j = \frac{1}{z^2}$, the update equation (8) becomes

$$\begin{aligned} \hat{g} &= \left(\zeta + D - z^2 \sum_{k=1}^{D-1} \frac{1}{z^2} \right)^{-1} \\ &= \frac{1}{\zeta + 1}. \end{aligned}$$

Solving for ζ gives us the critical value $\zeta = z^2 - 1$.

Although this distribution is a fixed point of (8), we need analyze its linear stability to be sure that it is an attractor. We can do this by considering the partial derivatives

$$\left.\frac{\partial \hat{g}}{\partial g_{j_k}}\right|_{g_{j_k}=1/z^2} = \frac{z^2}{(\zeta+1)^2} = \frac{1}{z^2}.$$

Now, at every iteration we compute $\hat{g}$ based on $D-1$ messages $g_{j_1}, ..., g_{j_{D-1}}$, so the distribution is stable so long as

$$\sum_{k=1}^{D-1} \frac{\partial \hat{g}}{\partial g_{j_k}} < 1,$$

meaning that on average it is sufficient for

$$\frac{\partial \hat{g}}{\partial g_{j_k}} = \frac{1}{z^2} < \frac{1}{\langle D-1\rangle}.$$

Thus, when $z > \sqrt{\langle D-1\rangle}$, the message distribution at $\zeta = z^2 - 1$ is a real-valued δ-function and L_z has no spectral density at this point. In the case of $G(n, c/n)$, both D and $D-1$ are Poisson-distributed with mean c, so the condition becomes $z > \sqrt{c}$.

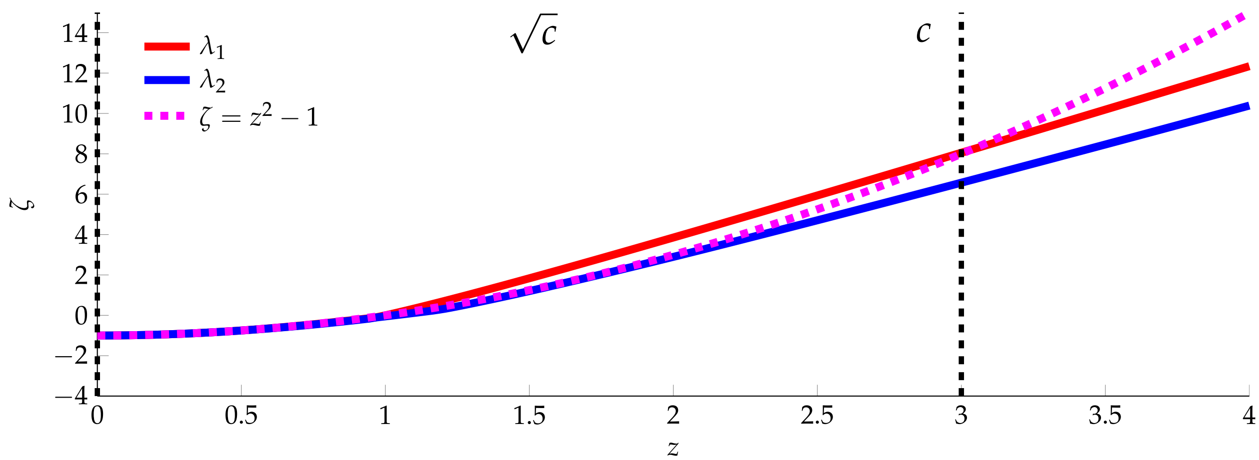


Figure 4: *First and second eigenvalues for L_z as a function of z for a single $G(n, c/n)$ graph with $n = 2000$ and $c = 3$. For $z \in [\sqrt{c}, c]$, the curve $\zeta = z^2 - 1$ appears to lie between λ_1 and λ_2.*

An Upper Bound for the Edge of the Bulk

In fact, even without an analytic expression for the fixed point distribution, we can compute an upper bound for the edge of the bulk. In particular, so long as D has an infinite tail, there is no spectral density for $\zeta > z^2 - 1$. First, if every g_{j_k} is confined to the interval $[0, \frac{1}{z^2}]$ on the real line,then

$$0 \leq \frac{1}{\zeta + D} \leq \left(\zeta + D - z^2 \sum_{k=1}^{D-1} g_{j_k}\right)^{-1} \leq \frac{1}{1-\zeta} \leq \frac{1}{z^2},$$

Thus for $\zeta > z^2 - 1$, there is a real-valued solution to (9) with support confined to this interval. As above, we can analyze the linear stability of this distribution by taking partial derivatives. To do so, we'll write $\hat{g}$

in terms if its real and imaginary parts as $\hat{g} = \hat{a} + i\hat{b}$, where

$$\hat{b} = \frac{z^2 \sum_{k=1}^{D-1} b_{j_k}}{\left(\zeta + D - z^2 \sum_{k=1}^{D-1} a_{j_k}\right)^2 + \left(z^2 \sum_{k=1}^{D-1} b_{j_k}\right)^2},$$

and $g_{j_k} = a_{j_k} + i b_{j_k}$. Now, taking partial derivatives at the real line,

$$\left.\frac{\partial \hat{b}}{\partial b_{j_k}}\right|_{b_{j_k}=0} = \frac{z^2}{\zeta + D - z^2 \sum_{k=1}^{D-1} a_{j_k}},$$

and since $a_{j_k} = g_{j_k} \in [0, \frac{1}{z^2}]$ and $\zeta > z^2 - 1$,

$$\left.\frac{\partial \hat{b}}{\partial b_{j_k}}\right|_{b_{j_k}=0} \leq \frac{z^2}{(\zeta+1)^2} \leq \frac{1}{z^2}.$$

Just as above, then, this distribution is stable so long as $z > \sqrt{\langle D-1 \rangle}$, or $z > \sqrt{c}$ in the $G(n, c/n)$ case where the degrees are Poisson distributed with mean c. Figure 4 compares $\zeta = z^2 - 1$ with λ_1 and λ_2 for a particular $G(n, c/n)$ graph; for $z \in \left[\sqrt{c}, c\right]$, $\zeta = z^2 - 1$ lies between the first and second eigenvalues.

Toward the True Bulk Edge?

Our numerical results indicate that for $G(n, c/n)$ the z-Laplacian in fact has zero spectral density well below $\zeta = z^2 - 1$ for $z > \sqrt{c}$. However, Figure 8 shows that the stable real-valued distribution is very different below this point; it is no longer confined to the interval $\left[0, \frac{1}{z^2}\right]$, but instead appears to have infinite support. Because of this, it is unlikely that a similar argument to above will suffice to locate the true edge of the bulk. Instead, we turn to a study of the real-valued fixed point distribution with the hope that it will give us insight into the true value for the edge of the bulk.

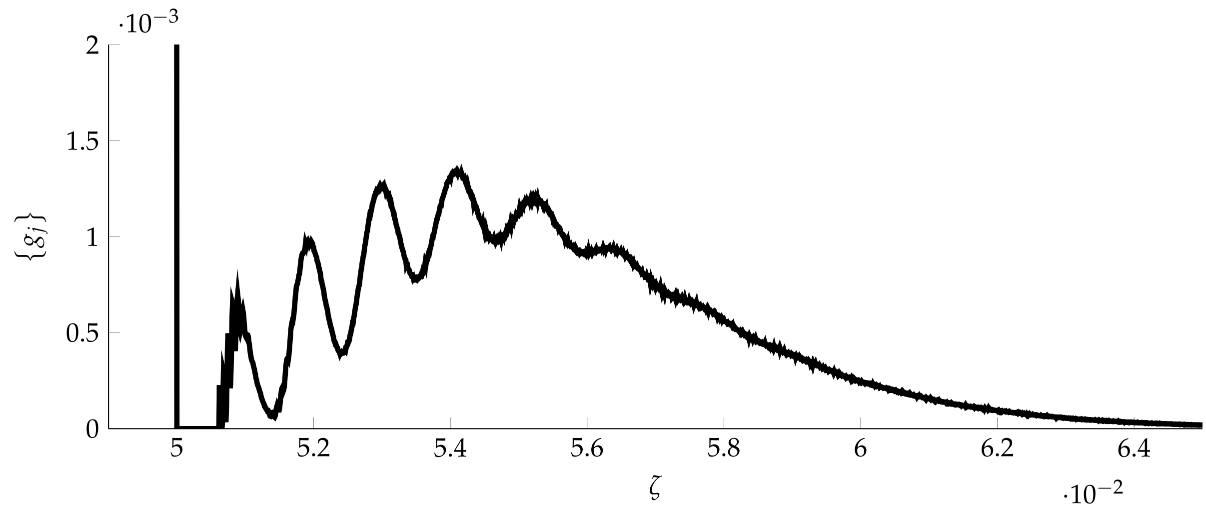


Figure 5: *Real fixed point message distribution,* $n = 10000$, $c = 3$, $z = c$, $\zeta = 18.9$; *computed from population dynamics iterations* $101 - 1000$.

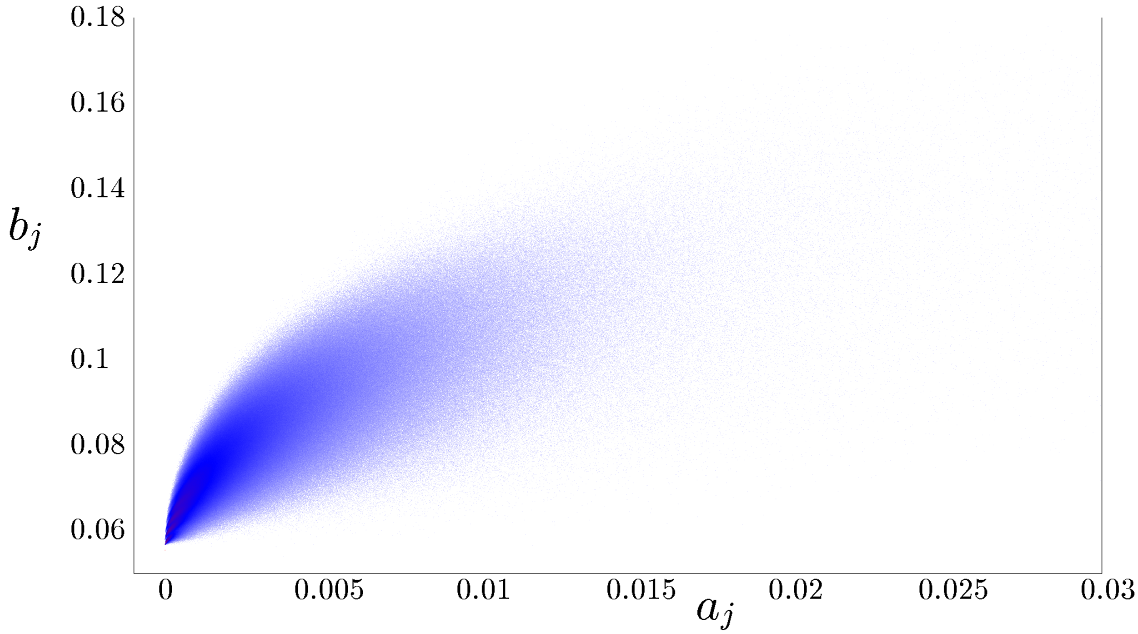


Figure 6: *Complex fixed point message distribution*, $n = 10000$, $c = 3$, $z = c$, $\zeta = 17.03$; *computed from population dynamics iterations* $101 - 1000$.

The Fixed Point Message Distribution

Following [1], we can think of the fixed point message distribution as defining a random variable $Y(\zeta)$ which satisfies

$$Y(\zeta) =_d \left(\zeta + D + \sum_{j=1}^{D-1} Y_j(\zeta)\right)^{-1}, \tag{9}$$

where D is distributed according to the degree distribution of the random graph ensemble, the $Y_j(\zeta)$ are i.i.d. copies of the random variable $Y(\zeta)$, and $=_d$ indicates that the left- and right-hand sides have the same distribution. Bordenave and Lelarge prove that an equivalent equation to (9) is guaranteed to have a solution under some mild assumptions [1].

A closer examination of (9) give some insight into the structure of the real-valued distribution in Figure 5. We can think of $Y(\zeta)$ as a discrete convolution with the degree distribution; in the case of $G(n, p)$, we have

$$Y(\zeta) = \sum_{k=0}^{\infty} \frac{\mathrm{Pois}_c(k)}{\zeta + k + \sum_{j=1}^{k-1} Y_k(\zeta)}, \tag{10}$$

where the kth term can be interpreted as the distribution of incoming messages to a vertex of degree k. Numerical results bear this out, and are captured in Figure 7, where we see that each of the humps in Figure 5 corresponds to a single term in (10).

Moreover, as in [1] we can derive an expression for the moment generating function of $Y(\zeta)$. Let $f(u, \zeta) = \mathbb{E}\left(e^{uY(\zeta)}\right) = \sum_{m=0}^{\infty} \frac{u^m \mathbb{E}Y(\zeta)^n}{m!}$ be the moment generating function; using the identity

$$e^{iuw} = 1 - \sqrt{u} \int_0^{\infty} \frac{J_1(\sqrt{ut})}{\sqrt{t}} e^{-itw^{-1}}\, dt,$$

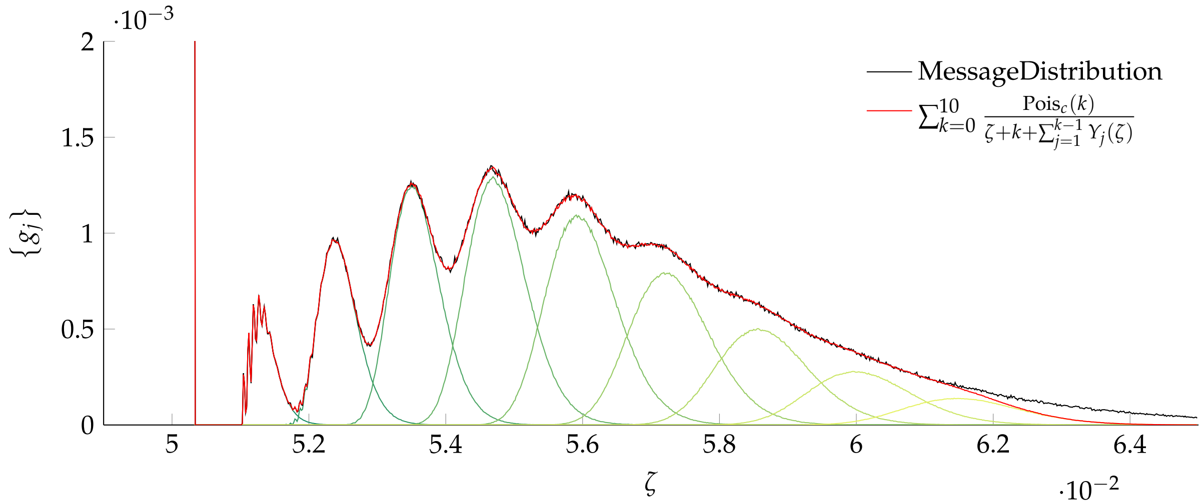


Figure 7: *Limiting message distribution from Figure 5 as a discrete convolution. Each green curve represents one term in* (10)*, i.e. the distribution of messages for a fixed degree k.*

we can write

$$\begin{aligned}
f(u,\zeta) = \mathbb{E}\left(e^{uY(\zeta)}\right) &= \mathbb{E}\left(e^{iu(Y(\zeta)/i)}\right) \\
&= 1 - \sqrt{u}\int_0^\infty \frac{J_1(\sqrt{ut})}{\sqrt{t}}\mathbb{E}\left(e^{-it(i/Y(\zeta))}\right)\,dt \\
&= 1 - \sqrt{u}\int_0^\infty \frac{J_1(\sqrt{ut})}{\sqrt{t}}\mathbb{E}\left(e^{t\left(\zeta+D+1-z^2\sum_{j=1}^{D} Y_j(\zeta)\right)}\right)\,dt \\
&= 1 - \sqrt{u}\int_0^\infty \frac{J_1(\sqrt{ut})}{\sqrt{t}}e^{t(\zeta+1)}\mathbb{E}\left(e^{tD}\prod_{j=1}^{D} e^{-tz^2 Y_j(\zeta)}\right)\,dt \\
&= 1 - \sqrt{u}\int_0^\infty \frac{J_1(\sqrt{ut})}{\sqrt{t}}e^{t(\zeta+1)}\phi\left(e^t f(-tz^2,\zeta)\right)\,dt,
\end{aligned}$$

where $\phi(x) = \mathbb{E}(x^D)$. As above, for $G(n,p)$ both vertex degree and excess degree are Poisson distributed with mean c; in this case $\phi(x) = e^{c(x-1)}$, giving

$$f(u,\zeta) = 1 - \sqrt{u}\int_0^\infty \frac{J_1(\sqrt{ut})}{\sqrt{t}}\exp\left(t(\zeta+1) + c\left(e^t f(-tz^2,\zeta) - 1\right)\right)\,dt.$$

We obtain an expression for the nth moment of $Y(\zeta)$, albeit one in terms of the moment generating function $f(u,\zeta)$, by taking

$$\begin{aligned}
\mathbb{E}(Y(\zeta)^n) &= \frac{\partial}{\partial u^n}\left[f(u,\zeta)\right]_{u=0} \\
&= -\int_0^\infty \frac{\partial}{\partial u}\left[\sqrt{\frac{u}{t}} J_1(\sqrt{ut})\right]_{u=0} e^{t(\zeta+1)}\phi\left(e^t f(-tz^2,\zeta)\right)\,dt \\
&= -\int_0^\infty \frac{\partial}{\partial u}\left[\sqrt{\frac{u}{t}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}\left(\frac{\sqrt{ut}}{2}\right)^{2m+1}\right]_{u=0} e^{t(\zeta+1)}\phi\left(e^t f(-tz^2,\zeta)\right)\,dt \\
&= -\int_0^\infty \frac{\partial}{\partial u^n}\left[\sum_{m=0}^{\infty}\frac{(-t)^m u^{m+1}}{m!(m+1)!2^{2m+1}}\right]_{u=0} e^{t(\zeta+1)}\phi\left(e^t f(-tz^2,\zeta)\right)\,dt \\
&= \int_0^\infty \left(\frac{(-t)^{n-1}}{2^{2n-1}(n-1)!}\right) e^{t(\zeta+1)}\phi\left(e^t f(-tz^2,\zeta)\right)\,dt. \qquad (11)
\end{aligned}$$

The above equation drives home the recursive, self-similar nature of the limiting distribution, in that each moment of the fixed point distribution depends on every other moment; we cannot even hope to obtain an analytic or closed form expression for any of the moments independent of an analytic expression for the distribution as a whole.

5 Conclusion

The z-Laplacian is a promising addition to the current toolbox of spectral community detection algorithms. L_z shares spectral properties with the non-backtracking operator that has been shown to vastly outperform other spectral protocols on sparse networks, and experimental observations of the z-Laplacian's spectrum indicate that it too should succeed in extracting community structure down to the universal detectability threshold [3],[4]. Although we have not yet produced a rigorous argument to this effect, our study of the spectral properties of L_z has employed message-passing and population dynamics approaches potentially usable on sparse random matrices more broadly, and we believe that continued study of the fixed point distribution of (9) will give us further insight into the spectrum of this novel operator.

Acknowledgments

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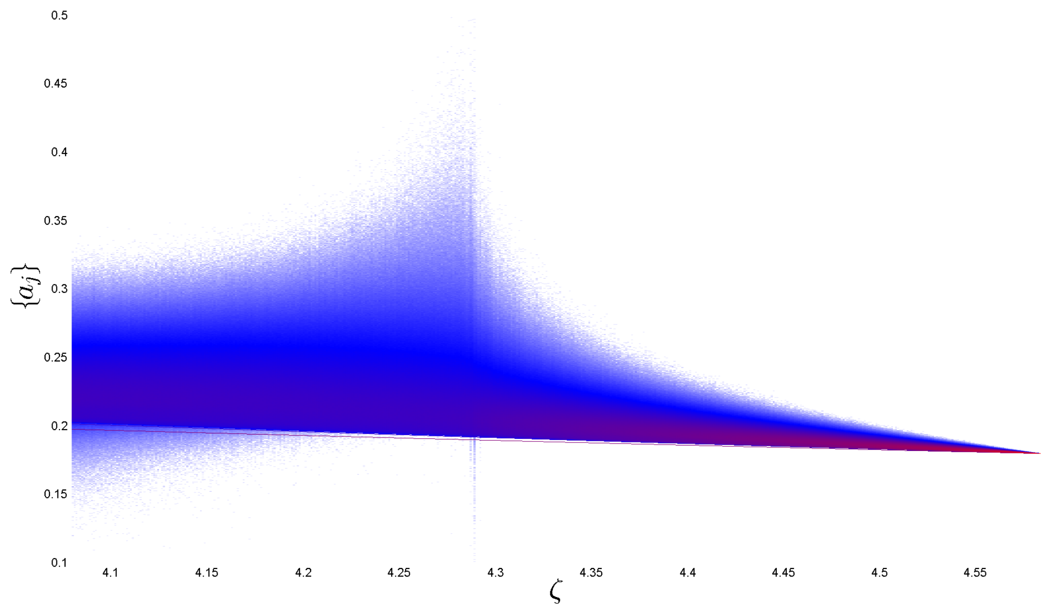
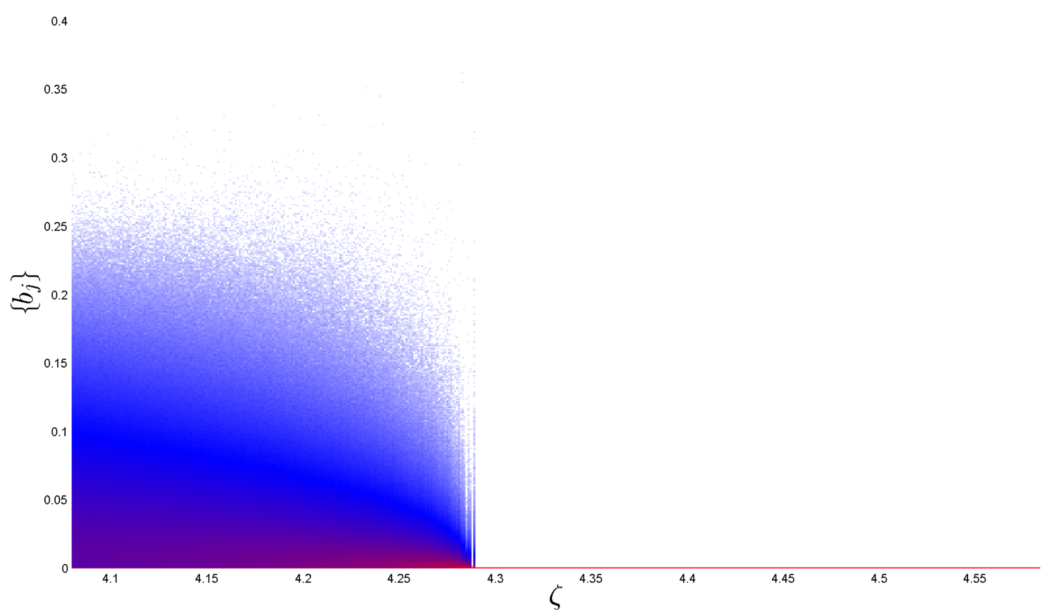


(a)



(b)

Figure 8: *Real (a) and complex (b) parts of the fixed point message distribution for $n = 1000$, $c = 3$, and $z = 2.36$ for ζ between 4.07 and $4.58 = z^2 - 1$. D was Poisson distributed with mean 3, and the distributions are taken over the 100th to 200th iterations. The stable real distribution persists well below $\zeta = z^2 - 1$.*