Chaos and Politics: Applications of Nonlinear Dynamics to Socio-Political Issues

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APPLICATIONS OF NONLINEAR DYNAMICS TO SOCIO-POLITICAL ISSUES

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Abstract

We discuss the extent to which recent improvements in our understanding of inherently nonlinear phenomena present challenges to the use of mathematical models in the analysis of environmental and socio-political issues. In particular, we demonstrate that the *deterministic chaos* present in many nonlinear systems can impose fundamental limitations on our ability to predict behavior even when precisely defined mathematical models exist. On the other hand, results from chaos theory can provide means for better accuracy for short-term predictions even for systems which appear to behave completely randomly. Chaos also provides a new paradigm of a complex temporal evolution with bounded growth and limited resources without the equivalent of stagnation and decay. This is in contrast to a traditional view of historical evolution which is perhaps best expressed by the phrase: "If something stops growing, it starts rotting". The exploration of a large number of states by a single deterministic solution creates the potential for adaptation and evolution. In the context of artificial life models this has led to the notion of *Life at the edge of chaos* expressing the principle that a delicate balance of chaos and order is optimal for successful evolution. Since our primary aim is didactic, we make no attempt to treat realistic models for complex issues but rather introduce a sequence of simple models which illustrate the increasingly complicated behavior that can arise when the nonlinearity is properly taken into account. We begin with the familiar elementary model of population growth originally due to Malthus and indicate how the incorporation of nonlinear effects alters dramatically the expected dynamics of the populations. We then discuss models which are caricatures of two issues – weather prediction and international arms races. Among the arms race models we consider a special class which is related to population dynamics and which was first introduced

by L.F. Richardson after WW I. The examples we discuss, however, have discrete time dependence. For certain ranges of their control parameters, these models exhibit *deterministic chaos* , and we discuss how this behavior limits our ability to anticipate and predict the outcomes of various situations.

We then briefly describe methods to exploit the high sensitivity of chaotic systems to dramatically increase the capability of both forecasting and control of chaotic systems. We show that many different solutions can coexist even in simple models and how machine learning methods such as neural nets and genetic algorithms can be used to find classes of optimal solutions.

We speculate on some generalizations of arms control models into object oriented frameworks which allow simultaneous modelling on different levels of quantitative formalizations: In a computational network we can have nodes which represent purely conceptual models of areas where quantitative analysis would be inappropriate and other nodes for which a hierarchical structure of models of arbitrary quantitative detail and sophistication can be generated.

Finally, we close with a few remarks on our general theme, stressing that, despite its limitations and because of its challenges, mathematical modeling of complex environmental and socio-political issues is crucial to any efforts to use technology to enhance international stability and cooperation.

1 I. Introduction

In a conference on "Impact of Chaos on Science and Society" it is natural to ask what contribution one of the fundamental *technologies* of modern science – namely, mathematical modeling, typically including extensive computer simulations – can bring to global efforts to enhance stability and cooperation. A moment's consideration makes it clear that the potential contribution is profound : for instance, one can immediately identify a number of environmental issues – the green house effect, ozone depletion, nuclear winter – of overwhelming international concern in which an accurate quantitative understanding of the causal relationships between specific local actions and global consequences can only be obtained from studies of highly sophisticated mathematical models containing many subtle and counterbalancing effects. The situation becomes even more complex in the modeling of socio-political issues – for example, economic consequences of exchange rate or stock market fluctuations, third-world debt, combatting the AIDS epidemic – when in addition to the complex technical issues one must try to account for the vagaries of human psychology. We expect that issues based on global ecological problems as well as general consequences of limited resources and degrading environmental conditions will become increasingly relevant factors which have to be taken into account in any arms race model.

Each of these issues involves many individual components, interacting with each other in complicated ways. Clearly one immediate, primarily technical, challenge to mathematical modeling is to quantify these interactions; for instance, in the *green house* issue, to understand quantitatively the role of the Brazilian rain forest in helping to stabilize the amount of carbon dioxide in the atmosphere. Such technical questions will – and should – remain the purview of experts, and in some cases the resolution of these questions requires the successful collaboration of experts from many different disciplines. In this area, the challenge is to develop well-defined models, properly reflecting the essential features of the problems; for this technical challenge, the appropriate caveat is perhaps best captured by the inelegant but celebrated phrase: "garbage in, garbage out".

Beyond this problem-specific technical challenge, however, are challenges and limitations that arise from the very nature of systems in which many elements, some of which may adapt their behavior in time, are interacting. Such *adaptive complex systems* often behave in ways that seem non-intuitive – or even counter-intuitive – based on everyday experience. For instance, the inherently nonlinear nature of these systems means that they can exhibit sudden and dramatic changes in the form of their behavior when small changes are made in the parameters describing the interactions within the system. Further, *emergent properties* – that is, characteristics whose existence is not at all apparent in the initial formulation of the system – frequently arise, and theories of *self-organization* in natural systems [31] have attempted to analyze certain aspects of this behavior.

While non-experts can hardly expect – nor be expected –to be aware of the subtle details surrounding the technical modeling of specific aspects of these complex issues, it is vital that in particular those responsible for making decisions on possible courses of action be aware of this second category of general constraints and characteristics that affect the applicability and reliability of the models. To achieve this awareness, it is essential to go beyond our conventional *linear* intuition and to develop an appreciation of what can –

as well as what can not – occur in complex adaptive systems. The development of the appropriate *nonlinear intuition* is extremely important, for it is clear that mathematical models, to the extent that they are credible, not only tell us what is likely to occur but *can limit our perceptions of what can occur*. Indeed, in our later discussion of the history of models of population growth, we shall exemplify this (potentially negative) aspect of modeling.

Our main goal in the present article is to assist in the development of this nonlinear mathematical *literacy* – or, in the current parlance, *numeracy* [55] – by describing and illustrating a few of the general challenges to and the limitations of mathematical modeling that arise from the complex, interacting, nonlinear nature of environmental and socio-political issues. Our perspective is based on the considerable recent progress that has been made in understanding nonlinear phenomena in the natural sciences [see, e.g., [8]. In particular, the surge of interest in nonlinear dynamical systems theory has shown that such concepts as *bifurcations, attractors, basins of attraction, deterministic chaos*, and *fractals* are essential for understanding the possible consequences of nonlinearity.

In the following article we shall try to introduce, primarily by example, most of these concepts. For definiteness and in recognition of our own limitations (particularly of time and space) we shall focus on two of the most important of these concepts: (1) *bifurcations*; and (2) *deterministic chaos*. For a discussion of these concepts we also have to introduce other notions like fractals, basins, and basin boundaries. *Bifurcations*, which are sudden (sometimes dramatic), qualitative changes in behavior of a system in response to small changes in the control parameters, are likely to be familiar to non-experts, primarily because many of the aspects of this phenomenon were discussed in the 1960's under the (perhaps overly pejorative) rubric of *catastrophe theory* [67]. Its relevance to the modeling of complex environmental issues is immediate: indeed, the primary concern associated with the *greenhouse effect* is that slight changes in the concentration of atmospheric carbon dioxide leading to a slight change in the average global temperature – perhaps as small as a degree or two Celsius – could lead to catastrophic changes in the climate (due to non-linear feedback mechanisms), inducing, for example, a man-made Ice Age. Of special interest here are *global bifurcations* which influence the structure of the *basin of attraction* of classes of solutions: The dependence of the future behavior of the system as a function of a current state, can depend very sensitively on the systems parameters. For one parameter a whole neighborhood of initial conditions can tend to the same asymptotic solution or attractor, i.e. all initial conditions were in the same *basin of attraction*. A small parameter change might move the *basin boundary* in a global, nonlinear way such that it splits the set of initial points into subsets which will approach different attractors. The geometric nature of this boundary can be very complex or *fractal* like it is in a coastline or in the coloring of marble. This can all happen even if the asymptotic state of the system is very orderly and not *chaotic*.

In contrast, the phenomenon of *deterministic chaos*, has only more recently begun to work its way into public awareness [28]; thus much popularization remains to be done before its full significance is appreciated. As indicated above, the essence of this phenomenon is that even in systems whose evolution from moment to moment follows precise deterministic laws, with no external random influences of any kind, the behavior over long times can be essentially unpredictable and in fact as random as a coin toss. That a system governed by deterministic laws can exhibit effectively random behavior runs directly counter to our normal intuition. Perhaps it is because this intuition is inherently linear; indeed, deterministic chaos *cannot* occur for linear systems. More likely, it is because of our deeply ingrained view of a *clockwork universe* , a view which in the West was forcefully stated by the great French mathematician and natural philosopher Laplace; in *Philosophical Essays on Probabilities*, Laplace wrote: "An intellect which at any given moment knew all the forces that animate Nature and the mutual positions of the beings that comprise it, if this intellect were vast enough to submit its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect, nothing could be uncertain; and the future just like the past would be present before its eyes." In short, Laplace argued that from knowledge of the initial state of the universe (and the forces) comes an exact knowledge of the final state of the universe. Indeed, in Newtonian mechanics, this belief is in principle true.

However, in the real world exact knowledge of the initial state is not achievable. No matter how accurately the velocity of a particular particle is measured, one can demand that it be measured more accurately. Although we may, in general, recognize our inability to have such exact knowledge, we typically assume that if the initial conditions of two separate experiments are *almost* the same, then the final conditions will be *almost* the same. For most smoothly behaved, *normal* systems, this assumption is correct. But for certain nonlinear systems, it is false, and *deterministic chaos* is the result.

At the turn of this century, Henri Poincaré, another great French mathematician and natural philosopher, understood this possibility very precisely and wrote (as translated in *Science and Method* [56]) :

"A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation *with the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the later. Prediction becomes impossible, and we have the fortuitous phenomenon."

In the physical sciences, the concept of deterministic chaos has already clarified a number of previously inaccessible phenomena, including for instance aspects of the transition to turbulence in fluids. An example relevant to our later discussion concerns the essential limits to the predictability of weather. Although weather prediction on the scale of a few days is currently accurate to an extent unimaginable a few decades ago, the theory of deterministic chaos tells us that we can not increase this prediction time much further, for the effort that would have to be expended on data collection and computer simulation grows *exponentially* with the desired forecasting time. This is a clear illustration of the type of quantitative limitations that deterministic chaos imposes on environmental modeling. But even in those cases in which chaos has not given us greater quantitative insight into how to model a specific process, it has broadened our view of the possible outcomes of models and deepened our understanding of the complexities of the behavior of nonlinear systems.

In the context of chaotic behavior of nonlinear dynamical systems we should discriminate between two categories: conservative systems which, for example, describe chaos in planetary motion, and dissipative systems, those which arise e.g. through self-organizing systems or driven and damped systems in general. In the latter case we are interested in situations, in which the chaotic behavior is attractive: there exists an open neighborhood of the chaotic set which is asymptotically (as time goes to infinity) attracted to it. The set of all points which is attracted to a specific attractor is called *Basin of attraction* of the same attractor.

To achieve our goal of exemplifying the consequences of nonlinear effects such as bifurcations and chaos, we shall take advantage of another recently understood fact about nonlinear systems, namely that some of their properties are universal: For example in the transition to chaos, basically three different routes have been identified and studied. These routes apply to a surprisingly large class of seemingly unrelated phenomena. Universality implies that the essential features of a particular behavior in a general nonlinear system can be deduced from studying the simplest system which exhibits this behavior. Thus instead of dealing with the extremely complicated models required to describe real environmental issues, we shall instead introduce and study a sequence of simple models which illustrate the increasingly complicated behavior that can arise when the nonlinearity is properly taken into account. We apologize in advance to mathematically sophisticated readers and trust they will accept our deliberately over-simplified presentation in the sense in which it is intended.

In Section II we begin with perhaps the most basic of all human environmental issues – namely, population dynamics – and present a very condensed historical perspective on the development of models for the growth of populations of single species[1] Starting from the familiar elementary model of (exponential) population growth originally due to Malthus, we show that the inclusion of nonlinear effects, such as limits to growth and the discreteness associated with species not having continuous reproduction, alters dramatically the expected dynamics of the populations. [2]

In particular, we show that in certain circumstances nonlinear models of population dynamics can predict periodic *cycles* in which the population oscillates in time. We also show that in other circumstances these same models can predict erratic, non-periodic population fluctuations: as we shall see, these fluctuations are described by deterministic chaos. We explore the origins of this behavior and attempt to develop an intuitive feel for

[1]The term *population dynamics* has to be understood in a very general sense: Besides populations in an ecological systems we can also refer to populations of chemical species or *populations of arms expenditures* etc. The basic mathematical equations are not affected by the context or interpretation.

[2]Although this model appears to be mathematically overly simplistic and anachronistic, it had an enormous impact on societies especially in economic theoretical thinking. It still leads to heated discussion as is described in [52]

its consequences.

In Section III, we turn to a problem, confronted on a daily basis by all of us, which is both of considerable importance to global security and a long-standing (and frustrating !) challenge to mathematical modeling: namely, weather prediction. We illustrate how the early work of L.F. Richardson exposed limitations on weather prediction, arising from inherent, nonlinear features characteristic of weather. We then discuss an explicit (and very simplified) model, proposed in the 1960's by E. N. Lorenz, which shows all the mathematical properties of a general nonlinear phenomenon, including bifurcations from one type of behavior to another and, for certain ranges of their parameters, deterministic chaos. We discuss how this behavior limits *in principle* our ability to predict the weather, essentially independent of the computational and observational resources available to us.

In Section IV, we turn to models of the arms race, again predicating our discussion on the pioneering work of L.F. Richardson. We develop a sequence of increasingly less unrealistic (double negative intended !) models to describe the competition among nations in this particular arena and illustrate some initially surprising similarities among these models and the models of population dynamics discussed in Section II. We illustrate two distinct consequences of the possibility of chaos in these models – namely, the existence of actual chaotic fluctuations in time in the expenditures on arms and defense and the uncertainty which chaos introduces even in simple *win/lose* situations – and discuss the implications of these consequences for both further modeling and decision making.

In Section V we present some speculations on the future outlook for the the development of *computer-aided politics* . We discuss situations for which optimal solutions are not known to exist and where fast, interactive computer-game interfaces, with modern visualization and audification tools, can likely give powerful heuristic insights into strategies for finding solutions which are close to optimal.

Finally, in Section VI, we close with a few remarks on our general theme, stressing that,ϒ despite its limitations and because of its challenges, mathematical modeling of complex environmental and socio-political issues is crucial to any efforts to use technology to enhance international stability and cooperation.

2 II. The Evolution of Simple Models for Population Dynamics

One of the first-studied and still most important environmental issues is that of population dynamics. Clearly the nature and rate of growth of the human population is of central interest to environmental planners, but so also are a number questions concerning the population dynamics of competing species, involving predator/prey, parasite/host, or exploiter/victim relationships. Further, at the elementary level, the mathematical models describing population dynamics are simple to describe and easy to motivate. Thus it is natural to use them as a means of introducing the nonlinear phenomena that we wish to discuss.

The simplest models of the population dynamics of a single species involve the evolution in time of a single variable which represents the total population of that species. Models describing the competitive or symbiotic relationships among several species will typically involve several variables, one for each of the individual populations. Despite the very abbreviated and selective history presented here, one can see in the development of these models essentially all the general features to which we referred in the introduction.

A. Malthusian Dynamics and Exponential Growth

At the end of the 18th century, in his *Essay on the Principle of Population*, Malthus [41] , based on the delicately phrased postulate that "the passion between the sexes is necessary, and will remain nearly in its present state" formulated the conclusion that "population, when unchecked, increases in a geometrical ratio." Although Malthus in fact recognized immediately that such unbounded growth would not be possible and discussed in his essay many checks on growth, nonetheless the model now inextricably linked to his name is that which describes the growth of an isolated system evolving continuously in time in a constant environment and facing no limits to growth. In the precise language of modern differential calculus, if one defines the population at time t to be $N(t)$ and notes that, by Malthus' postulate, the rate of change in time of the population – denoted by $\frac{dN}{dt}$ – is proportional to $N(t)$ itself, one finds

$$\frac{dN(t)}{dt} = rN(t). \tag{1}$$

In this equation r is a *constant* which represents the *fecundity* (i.e., the efficacy of reproduction) of the population; clearly it is an essential parameter, in that it entirely determines that dynamics of the population. Indeed, the solutions to (1) can be given explicitly and their nature is determined by the value of r. If the population at a time $t = 0$ has an initial value $N(0) = N_0$, the size of the population at any later time t is given by

$$N(t) = N_0 e^{rt}. \tag{2}$$

The solutions represent either unbounded exponential growth (for $r > 0$, see Fig. 1) or exponential decay ($r < 0$, see Fig. 2). For the particular case of $r = 0$, the population remains constant.

Viewed as a dynamical system, the *Malthus equation* (1) is extremely simple. It is a linear – the variable $N(t)$ occurs only to the first power in the equation, so that adding together two different solutions to (1) still gives a solution – and, as (2) shows, can be solved analytically exactly for all time and for all initial conditions. For general r, there is only one *fixed point* – i.e., a value of $N(t)$ such that the $\frac{dN(t)}{dt} = 0$, so that population that does not change in time – and that is $N(t) = 0$: no population at all !

Although (1) is linear, the parameter value $r = 0$ is a point of *bifurcation* in the sense discussed in the introduction, for there is a fundamental change in the nature of the solutions at that point. However, as we shall see, the nature of the change in behavior is very much simpler than complex bifurcation structure observed in typical nonlinear systems. Note also that, except at $r = 0$, the solutions are stable in the sense that their nature does not change for small changes in r.

The Malthus equation is an excellent example of our earlier observation that the conclusions derived from a model can control our perceptions not merely of what is *likely* but also of what is *possible*. Malthus' work had enormous influence; indeed, the term *Malthusian* still retains its implication of inexorable and destructive growth. The idea that population growth was inexorable and eventually had to lead to crisis had profound influence on both the evolutionary theories Darwin and the economic theories of Ricardo.

In view both of its wide application and of our present broader perspective that makes the Malthusian predictions for growth seem rather naive, it is important to stress that Malthus equation actually often works well in regimes in which limitations to growth are not yet effective. A very relevant example is the initial phase growth of epidemics, which typically does follow the exponential form. Indeed, the *absence* of the expected exponential growth during the initial phases of the AIDS epidemic proved to be a key observation in the development of a mathematical model consistent with the true behavioral patterns of the various risk groups [13]. However, if extended beyond its region of validity, the naive exponential growth can lead to ridiculous results: an example is the amusing prediction, made in the last century in connection with an estimate the traffic development in New York City, that based on the rate at which the number of horse carriages was increasing, within the then foreseeable future all streets in New York would be covered by several feet of horse manure !

B. Continuous Logistic Dynamics and Limits to Growth

Two generations after Malthus' Essay, the Belgian social statistician Pierre-Francois Verhulst formulated and named the now-celebrated *logistic* equation to model the effects of limits to Malthusian growth of the population. Although there are many sources of limits to population growth, a fundamental one is the *carrying capacity* of the environment: namely, that if the population increases beyond a certain size, the environment is unable to support it, and the resulting effective reproduction rate or *fecundity* becomes negative. One can model this effect by replacing the constant r in eqn. (1) by an expression of the form $r(N) = r(1 - N/N_\infty)$. Notice that this expression for $r(N)$ approximates the constant value r of the Malthus equations in the limit when populations are much smaller than N_∞; in this region exponential growth is quite reasonable. As the population grows, however, a point is reached beyond which the impact of the population on its environment can no longer be neglected and, in fact, there exists a maximal sustainable population size N_∞ which will be approached asymptotically.

The specific form of the (continuous time) logistic equation is

$$\frac{dN(t)}{dt} = rN(t)(1 - \frac{N(t)}{N_\infty}). \tag{3}$$

The logistic equation is clearly nonlinear – note the presence of term proportional to $N(t)^2$ – and possesses *two* fixed points : $N(t) = 0$ and $N(t) = N_\infty$. Considering only the physically relevant case of positive intrinsic fecundity $r > 0$, we find that the fixed point at $N = 0$ is *unstable* – that is, solutions perturbed slightly from it grow in time – whereas that at $N = N_\infty$ is *stable*. Despite the nonlinear nature of (3), since it involves only a single variable we can again find an analytical solution, which for the physically relevant case $r > 0$ is given by:

$$N(t) = \frac{N_0}{N_0 + (N_\infty - N_0)e^{-rt}}. \tag{4}$$

In Fig. 3 we plot several solutions of the logistic equation for different initial conditions. Note that for initial values larger than the equilibrium solution, the population *decreases* until it reaches the stable fixed point value of N_∞, whereas for smaller initial values the population *increases* to this value. Further, changes in the parameter r will induce no quantitative changes in the long-time dynamics of the system. Therefore robust predictions can be made: if there is any population at all, it has a unique long-term future, with the asymptotic population given by N_∞. As we shall see below, this strong predictability is not expected in more realistic models.

C. Discrete Logistic Dynamics and Deterministic Chaos

An important feature of the population dynamics of many species [43] is that reproduction is episodic rather than continuous in time: for certain insects, for example, the population in a given year can be expressed as a function of the population in the previous year. Mathematically, this means that the continuous-time, differential equation (3) above is replaced by a discrete-time *map* , which relates the population at one discrete value of the time to that at the next discrete value. This seemingly minor change in the mathematical structure of the equations has a profound impact on the possible modes of behavior of the population.

To underscore this qualitative difference, let us change notation by labeling the discrete sequence of times by the integer n and using the variable x_n to denote the population at *time n*. If for simplicity we scale the carrying capacity of the environment to the value 1, then the discrete logistic *map* takes the form

$$x_{n+1} = rx_n(1 - x_n). \tag{5}$$

One can show that in order that the population never become negative, the values of x_n must be restricted to $0 < x_n < 1$ and the allowed values of the *fecundity* r must satisfy $0 < r \leq 4$. The development of the population is determined by iterating the map as many times as desired, and one is particularly interested in the behavior as *time* – that is, n, the number of iterations, approaches infinity. Specifically, if an initial population is picked at random in the interval (0,1) and iterated many times, what is the behavior of the population after all the *transients* have died out ?

It turns out that this behavior depends critically on the fecundity parameter and exhibits in certain regions *bifurcations* – the sudden and dramatic changes in response to small variations mentioned in the introduction – in r. For $0 < r < 1$, one can show that the value of x_n drops to 0 as n approaches infinity no matter what its initial value. In other words, after the *transients* disappear all points in the interval (0,1) are *attracted* to the *attracting fixed point* at $x = 0$. By standard mathematical techniques, the (linear) stability of this fixed point can be studied [24] and one can show that for $r < 1$ this fixed point is stable. Thus these low values of fecundity are insufficient to sustain any population at all, and the species dies out.

At precisely $r = 1$, there is a bifurcation, and the fixed point at $x_n = 0$ becomes unstable and is replaced by another fixed point, the value of which depends on r. This value is readily calculated, since at a fixed point $x_n = x_{n+1} = x^*$; substituting this into equation (5), we find

$$\lim_{n \to \infty} x_n = x^*(r) = 1 - \frac{1}{r}. \tag{6}$$

Hence as the value of r moves from one towards three, the value of the fixed point x^* moves from zero toward two-thirds. In population terms, the species approaches a fixed final population; this is just like the behavior observed above in the *continuous-time* logistic model for population dynamics, apart from the additional subtlety that for eqn. (5) the asymptotic value of the population depends on the fecundity. Again, a stability analysis shows that this fixed point is stable in the range $1 < r < 3$.

A more interesting bifurcation occurs as $r = 3$. Suddenly, instead of approaching a single fixed point, the long-time solution oscillates between *two* values: the model has an *attracting limit cycle* of period 2 and predicts that the asymptotic population will undergo *biennial* cyclic variations ! As one further increases r, the behavior of the asymptotic population becomes incredibly complicated, as shown in Fig 5, which depicts the attracting set – that is, the asymptotic population – in the logistic map as a function of r. Here we see clearly the bifurcation at $r = 3$ to the period two limit cycle. But more strikingly, as r moves toward 3.5 and beyond, there are also first period 4 and then period 8 limit cycles and then a region in which the attracting set becomes incredibly complicated. In fact, if one analyzes the logistic map carefully, one finds that as r is increased the after the period 8 cycle come cycles with periods 16, 32, 64, and so forth, with the process stopping only as the period goes to infinity. All this occurs in the *finite* region of r below the value $r_c = 3.56....$. This (infinite) sequence of bifurcations is, quite naturally, termed a *period-doubling cascade* , and is now recognized as one of the classic routes from regular to chaotic behavior. At many (but not all, as the Fig 5 shows) values of r above r_c the attracting set shows no apparent periodicity whatsoever and indeed consists of a sequence of points x_n that never repeats itself. For these values of r, the simple logistic map exhibits *deterministic chaos* and the resulting attracting set–far more complex than the attracting fixed points and limit cycles seen below r_c– is called a *strange attractor*.

Although one can develop a more formal mathematical description, the essence of deterministic chaos as be appreciated by noting several distinct features of this behavior. First, for the values of r where the motion is chaotic, there is exactly the *sensitive dependence on initial conditions* on initial conditions that was anticipated by Poincaré; if two initial populations are chosen very close to each other, the distance between their successive iterates diverges exponentially and after a short time the resulting population histories become as different as two separate coin-toss sequences. Second, the sequence is not only non-periodic but also does not consist of any finite number of different periods superimposed; in the appropriate mathematical jargon, the Fourier transform of the time series – which time series is shown in Fig 4 – does not have a series of distinct peaks corresponding to discrete frequencies but instead consists of a broad band of frequencies, a form traditionally associated with random external *noise*.

In view of the complexity of the attracting sets above r_c, it is not at all surprising that this model, like the typical problem in chaotic dynamics, defied direct analytic approaches. There is however, one elegant analytic result that quantifies the sensitive dependence on, and loss of information about, initial conditions that characterize deterministic chaos. Consider the particular value $r = 4$. Letting $x_n = \sin^2\theta_n$, then for $r = 4$ the logistic map can be rewritten

$$\begin{aligned} \sin^2\theta_{n+1} &= 4\sin^2\theta_n\cos^2\theta_n \\ &= (2\sin\theta_n\cos\theta_n)^2 \\ &= (\sin 2\theta_n)^2 . \end{aligned} \tag{7}$$

Hence the map is simply the square of the doubling formula for the sine function, and we see that the solution is $\theta_{n+1} = 2\theta_n$. In terms of the initial value, θ_0, this gives

$$\theta_n = 2^n\theta_0. \tag{8}$$

This solution makes clear, first, that there is a very sensitive dependence to initial conditions and, second, that there is an exponential separation from adjacent initial conditions. By writing θ_n as a binary number with a finite number of digits– as one would in any digital computer– we see that the map amounts to a simple shift operation. When this process is carried out on a real computer, round-off errors replace the right-most bit with garbage after each operation and each time the map is iterated one bit of information is lost[1]. If the initial condition is known to 48 bits of precision, then after only 48 iterations of the map no information about the initial condition would remain. Said another way, despite the completely deterministic nature of the the logistic map, because of the exponential separation of nearby initial conditions, except for very short times, all information about the motion is encoded in the initial state, and none is in the dynamics themselves.

[1]In many computers the shift operation replaces the right-most bit by a zero which then leads to a spurious globally attracting fixed point at $\theta = 0$.

More mathematically, it can be shown that for $r = 4$ the logistic map leads to an asymptotic population which covers the whole interval of possible values of x densely, with a frequency distribution of points $\rho(x)$ in the allowed interval $0 < x < 1$ given by $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. Further, the transformation we performed for r =4 shows that the logistic equation is equivalent to a *Bernoulli process* generated by a bit shift operation. This process is the basis of many modern (pseudo)random number generators used in computers. Finally, for $r = 4$ we can make precise the statement that the deterministic dynamics of the logistic map leads to behavior as "random as a coin toss" . Let us divide the unit interval into two parts and assign the symbol H to all values $x_n < 0.5$ and T to all values $x_n > 0.5$. If we record only the sequence of H's and T's then for this value of r we cannot distinguish the logistic dynamics from a true coin-toss.

There is still much more that we can learn from this simple example, but lack of space compels us to be brutally brief in the descriptions of the following results. First, one can ask how typical for more general nonlinear systems is the *period-doubling cascade* that marked the transition from regular to chaotic behavior in the logistic map. It turns out that this *period doubling cascade* is by no means to only manner in which dissipative nonlinear systems move from regular motion to chaos. Many other routes – among them *quasi-periodic* , *intermittent*, and so forth– have been identified and analyzed.

Second, if, however, the period-doubling cascade is observed in a system, it is possible to argue theoretically that there are *universal* features of this cascade that do not depend on the specific nature of the logistic map; this very important theoretical result concerning *universality* in nonlinear dynamics was confirmed by the observation of these universal features of period doubling in an actual fluid experiment. We shall later refer to an illustration of this universality in a model of weather prediction.

Third, Fig. 5 illustrates one additional obvious feature of the attracting set of eqn (5): namely, that it contains non-trivial– and, in fact, self-similar– structure under magnification. Indeed, in the mathematical model this self-similar structure occurs on all (smaller) scales; consequently, this figure is one example of a class of complex, infinitely ramified geometrical objects called *fractals* that typically arise in chaotic dynamical systems. Importantly, this fractal structure occurs in the space of parameters – here r – of the problem. There are also important examples where fractals are observed in the space of initial values of the problem.

Fourth, whereas in both the continuous-time logistic dynamics and the logistic map there exists for any given value of r only one attractor, in more general nonlinear dynamical systems one can find multiple attractors and asymptotic modes of behavior. In these cases, the initial conditions from which the evolution starts will determine to which final attractor the system will evolve asymptotically. The set of all initial configurations which lead to the same attractor is called *basin of attraction* for that attractor. One can show that the boundaries between these basins of attraction can also have the very complicated *fractal* structure discussed above [29]. In that event, the system will exhibits another form of *sensitivity to initial conditions* in the sense that the attractor to which the system evolves, depends very sensitively on the precise initial condition. Note that this is true even if the attractor itself is not chaotic; a well-known example is a roulette wheel, where the final state is always a fixed point – one of the numbers on the wheel – but which fixed point depends, to the grief of many, very sensitively on the initial conditions. Note also that if the system does have multiple attractors, transitions between these attractors can be induced by external fluctuations. In many cases it turns out that there exists a very large variation in the robustness of these attractors against external noise: that is, some are considerably more stable than others. This range of stability can mean that in realistic environments only a few of the possible attractors are actually observed. In particular, as a result of this noise sensitivity, one can observe [45] the disappearance of chaos via a *noise-induced order* .

Fifth, within the region $r_c < r < 4$, the logistic map exhibits a number of more subtle phenomena associated with chaos. At $r = r_c$, the actual type of deterministic chaos exhibited by the map is known as *noisy periodicity* . In noisy periodic chaos there exists a periodic oscillatory pattern with a small scale chaotic modulation of the amplitude. In fig. 5 these solutions appear as chaotic bands. The chaotic bands are interspersed with periodic windows, which typically are quite sensitive to external fluctuations. The phenomenon is that the increase of a given parameter (the fecundity, in our example) leads to chaotic solutions but then a further increase of the same parameter suddenly restores orderly motion. The behavior associated with this return of regularity is named *intermittency* after a related phenomenon observed in fluid turbulence. Intermittent chaos exhibits an irregular alternation of ordered behavior and chaotic bursts. The duration of such regular motion can become extremely long such that the system appears to be non-chaotic until a surprising burst of chaos reveals the underlying complexity. In the context of environmental processes intermittency-type phenomena, basically non-reproducible failures of the standard behavior of the system, seem not to be too rare. The inverse phenomenon is also observed: namely that a seemingly robust chaotic behavior suddenly decays to a regular periodic attractor or a fixed point in what is technically known as a *crisis* event [29].

Finally, in view of the prominence of continuous-time models, it is important to indicate the level of complexity that they must contain in order to exhibit chaos. For systems involving only two coupled first-order differential equations – for example, simple population equations describing competing species such as predator/prey or host/parasite – the most complex behavior that can occur is a limit cycle, analogous to those seen in the logistic map for $3 < r < r_c$. However, when one considers three coupled first-order differential equations, then one can observe deterministic chaos. In the next section, we shall give an example of this in the context of modeling weather forecasting.

3 III. Predicting the Weather: An Intuitive Example of Chaotic Dynamics

We turn now to a discussion of a nonlinear model for a thoroughly familiar phenomena – weather prediction – which is quite significant for global security and survivability. Although the model discussed in the ensuing subsections can *not* be taken as a realistic model of the phenomena it caricatures, it *can* be viewed as a *half-way house* between the very simple models discussed above and actual models of complex environmental phenomena. Its value is thus indicating that the effects of nonlinearity – including bifurcations and deterministic chaos – are indeed likely to occur in more realistic models.

A. Richardson's View of the Mathematics of Weather Prediction

B. Weather Prediction and the Lorenz Equations

In the early 1960s Edward Lorenz, a meteorologist at MIT, began a series of investigations aimed at establishing that unpredictability in weather forecasting was not due to any external noise of randomness but was in fact compatible with a completely deterministic description. In a sense, Lorenz [39] was quantifying another qualitative insight of Poincaré, who in another prescient comment – all the more remarkable for its occurring in the paragraph immediately following our earlier quote from *Science and Method* – captured both the importance and the difficult of meteorology:

" Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance, so that many people think it quite natural to pray for rain or fine weather, though they would consider it ridiculous to ask for an eclipse by prayer ? We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and this is the reason why it all seems due to the intervention of chance. Here, again, we find the same contrast between a very trifling cause, inappreciable to the observer, and considerable effects, that are sometimes terrible disasters."

To demonstrate this sensitive dependence quantitatively, Lorenz began with a simplified model approximating fluid convection in the atmosphere. By expanding this model in (spatial) Fourier modes and by *truncating* the expansion to the lowest three modes, explicitly ignoring higher couplings, Lorenz obtained a closed system of three nonlinear first-order differential equations:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= -y + rx - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned} \tag{9}$$

In the application to atmospheric convection, x measures the rate of convective overturning, and y and z measure the horizontal and vertical temperature variations, respectively. There are three control parameters in (9) – σ, r, and b – which can be related to physical characteristics of the flow. Obviously, the nature of the solutions to eqn (9) is likely to depend strongly on the values of these parameters. We shall focus on a few illustrative cases. For simplicity, we fix values of the parameters $\sigma = 10$ and $b = 8/3$, and discuss several values of r [66].

For values of $r < 1$, all initial conditions for the Lorenz equations eventually evolve to a fixed point at the origin, $(x = y = z = 0)$. This is directly analogous to the behavior of the discrete logistic map for small values of its parameter. For $r > 1$, the fixed point at the origin becomes unstable, but for r less than about 24 all initial conditions are attracted to one of two coexisting fixed points: either $C_1 = (x = (\frac{8}{3}(r-1))^{\frac{1}{2}}, y = (\frac{8}{3}(r-1))^{\frac{1}{2}}, z = r-1)$ or $C_2 = (x = -(\frac{8}{3}(r-1))^{\frac{1}{2}}, y = -(\frac{8}{3}(r-1))^{\frac{1}{2}}, z = r-1)$.The two basins of attraction of these fixed points change in complicated ways as r is increased , but no other attractors appear for $r < 24$. For larger values of r, the nature of the attracting set itself changes dramatically, and one finds the celebrated *Lorenz (strange) attractor* , three perspectives of which are shown in Fig 6. This attractor represents a solution to eqn (9) which is contained completely in a bounded region of the $\vec{x} = (x, y, z)$ space but which never comes back on itself. Since solutions of differential equations correspond to continuous curves, the solutions to the Lorenz equations $\vec{x}(t) = (x(t), y(t), z(t))$ to be defined at all times $-\infty < t < \infty$ must be of infinite length, if they are not asymptotic to a closed curve. If the full attractor, generated by the infinite time series of points $(x(t), y(t), z(t))$ were plotted, one would see the trajectory looping around forever, never intersecting itself and hence never repeating. The exquisite filamentary structure seen in Fig 6 would exist on all scales, and even in the limit of infinite time the attractor would not form a solid volume in the (x, y, z) space. Again, this is a characteristic of *fractals* , for which the concept of dimension is generalized to non-integer values, expressing the fact that these objects in some sense lie *between* regular Euclidean objects such as ordinary lines, surfaces, and solids. (see e.g. [[42], [47]]). In the case of the Lorenz attractor, the fractal dimension is approximately $D = 2.07$.

If one continues to change the value of r, numerous additional bifurcations occur. Some involve the re-appearance of regular motion and, in particular, one can find examples of the *period-doubling cascade* to chaos (in particular, in the region just above $r = 215$). Importantly, as we anticipated in our discussion of *universality* , many characteristics of this cascade in the Lorenz equations are identical to those seen in the logistic map. Another less rigorous and more qualitative aspect of this universality is that similar equations have been shown also to describe approximately chaotic behavior of certain types of lasers (see e.g. [31]).

From the perspective of the original weather prediction problem, the existence of the strange attractor solution to the Lorenz equations implies that for a wide range of parameters the long-time behavior of the model simply can not be predicted, for the sensitive dependence on initial conditions means that two points starting very near each other on the attractor evolve in time in dramatically different ways. In popular language, this is referred to as the *Butterfly Effect* [28]: whether a butterfly in Beijing flaps its wings three times or only twice can (in principle) alter totally the weather in San Fransciso some days hence. The existence of deterministic chaos and strange attractors thus clearly indicates one type of limitation to mathematical modeling of nonlinear systems ranging from dripping faucets [64] through electronic devices and chemical reactions [31] to climatic oscillatory phenomena like *El Niño* [69] and perhaps even stock market dynamics (see e.g. [62]).

4 IV Chaotic Dynamics and Arms Race Models

The pioneering work of Richardson [58] set the stage for subsequent attempts to analyze quantitatively many questions of strategic military and economic competition between – and among – nations. In this section we discuss several simple and discrete variants of Richardson's equations [49], primarily as brief exercises to indicate how one goes about formulating and analyzing nonlinear models of socio-political issues. We should stress at the outset of this particular discussion that in our opinion the motivation of using simple, nonlinear dynamical models in the socio-political arena is somewhat different from that for using classical game theory where one was interested in finding an optimal strategy as the unique solution to a given problem. One of the essential lessons of chaos theory is that individual solutions (histories) are basically non-reproducible and therefore of very limited relevance. A system might have many different but equivalent solutions to the same problem.

In his original work, [58] Richardson developed a set of equations that applied to the arms race between *two* nations having just *one* variable describing each nation with the resulting two variables evolving in *continuous* time. As we have mentioned above, this assumption leads, on very general mathematical grounds, to extremely restrictive and non-typical theoretical consequences: namely, solutions to the original Richardson equations can not exhibit bounded behavior more complex than fixed points or periodic oscillations. However, as we have also indicated above, if one adds more variables – e.g., incorporating additional nations, or using several variables to describe each nation – or if one adds time-delays in the response of one nation to another's actions –e.g., the development or even deployment of a new weapons system is detected and responded to only after a certain period of time – then these more complex continuous time models can (and generally do) exhibit deterministic chaos. Since our primary purposes is an illustrative rather than a definitive description of modeling of the arms race, we adopt the simpler procedure, motivated above, of considering models in *discrete* time. In the case of an arms race model, this is in fact not such an unreasonable approach, for the discrete time can represent, for example, the fact that budget decisions are typically approved on an annual basis, or relevant information is not provided continuously. In this discrete time case, provided the underlying equations are nonlinear, one can observe already in the case of two nations all the striking features found in the chaotic nonlinear dynamical systems discussed above.

A. Two-Nation Richardson-type Models

In order to present clarify the process of constructing a simple discrete time Richardson model, let us consider the following example of two competing nations: Let $x_\alpha(t)$ be variables describing the arms expenditures at time t of nation α, where $\alpha = 1, 2, ..., d$ indicates which of the "d" nations is being considered. The number of nations (the *dimension* d) can be small, e.g., $d = 2$ if only NATO and WTO are considered or if only the USA and the USSR are considered. In the latter case, if Europe and China are taken additionally and independently one has $d = 4$. In the various local conflicts other countries are involved, but the dominant variables are always only a few. In physical systems the $x_\alpha(t)$ are sometimes called *modes* or *amplitudes* , or *order parameters* (however paradoxical this might sound when the behavior of $x_\alpha(t)$ is chaotic !).

How does $x_\alpha(t)$ behave as a function of time? We are most acquainted with stationary states, or *fixed points* , i.e., $x_\alpha(t) \to x^*_\alpha$ (a *fixed* constant) for increasing time t. The stationary state x^*_α depends, of course, on certain control parameters a – the analogues of the parameter r in the logistic map, eqn. (5) – which in this context express policies, strategies, economical constraints, etc. We usually explore first how the steady state $x^*_\alpha(a)$ depends on a and then alter the control parameters to adjust the state according to our demands and aims (or wishes). Stationary states, if stable, make life calculable and *safe* .

A.1 A Two-Nation Logistic-type Model

The appearance of chaotic solutions has been described in a discrete Richardson-type model [60] with the very simple form:

$$\begin{aligned} x_{t+1} &= 4ay_t(1-y_t) \equiv f_a(y_t) \\ y_{t+1} &= 4bx_t(1-x_t) \equiv f_b(x_t) \end{aligned} \tag{10}$$

where we have introduce the notation $x = x_1$ and $y = x_2$ to avoid notational complications with subscripts and time values. The variables x and y denote the respective fractions of the available resources which countries X and Y devote to armament.

To argue for the form of equations (10), we note that it is reasonable to assume a country's armament x_{t+1} is largely determined by the hostile armament y_t of the previous year; perhaps less reasonable is the factor $(1-y_t)$, since it reduces the armament efforts of a responding nation to 0 if the hostile country has the largest possible ($y_t = 1$) armament.

The model mentioned above is *basically one*-dimensional with two-year steps, $x_{t+2} = f_a(f_b(x_t)) \equiv f_{a,b}(x_t)$, and similarly for y_{t+2} with $f_{b,a}(y_t)$. Both countries are simultaneously in a steady state $x^*, y^*(= f_b(x^*))$, simultaneously bifurcate to a periodic state, or together enter chaos.

In [60] the transition to chaos was associated with unpredictable behavior, crisis-unstable arms races, and therefore with an increased risk for the outbreak of war. Although this interpretation can be disputed (see e.g. the discussion in [35], [30], it has become clear that the possibility of transitions to chaotic behavior must be taken into account in arms race models.

A second interpretation of the role of chaos in arms race models has been discussed in [30]: bounded, small-scale chaos can be part of a stable arms control situation. There although the other side's response cannot be anticipated in detail (possibly because of internal political problems of one country), the stability of the attractor itself allows for confidence in the fact that no disastrous surprises will occur. This would be expected if there exists a basic consensus within one nation about defense policy, but details of the budget are subject to internal discussions. The effect of couplings between internal opinion formation (between "hawks" and "doves") and interactions between two nations has been discussed in [2].

The appearance of this type of small scale chaos has to be distinguished from large chaotic fluctuations which could lead to configurations in which *crises* could lead to unbounded arms races, war, or economic collapse. Here crises are understood as sudden changes in the global behavior of the model: solutions can explode and attractors can change their size and locations. Close to such crises, a system is very sensitive to noise. (see e.g. [51] for a technical definition and discussion)

A.2: Resource-Limited Response

A similar arms race model for which the nonlinearity can be interpreted as an *economic constraint* has been discussed in [30]. Mathematically, this model is defined by the equations:

$$\begin{aligned} x_{t+1} &= x_t - k_{11}(x_t - x_s) + k_{12}y_t(1 - x_t) \\ y_{t+1} &= y_t - k_{22}(y_t - y_s) + k_{21}x_t(1 - y_t), \end{aligned} \tag{11}$$

where again x_t and y_t denote the fraction of the available resources devoted to armament in the countries X and Y during the year t. The *change* of armament or military expenditure, $x_{t+1} - x_t$, is taken to be proportional to the military effort y_t of the other country. The rate factor was considered as constant by Richardson. Here one assumes instead that this factor itself depends on the still-available resources and put $k_{12}(x_{max} - x_t)$. Just as in the population models discusses in Section II, the largest available fraction of resources, which we have called $x_{max} here$, can be taken to be 1 simply by re-scaling variables in equation (11). The constants k_{12} and k_{21} (both positive) correspond to the *defense intensity* of Richardson [58]. The constants x_s, y_s are the countries' *natural* self-establishing armament levels, corresponding to Richardson's *grievance* variable [58]. Finally, the constants k_{11} and k_{22} are the (positive) expense coefficients forcing the approach to x_s, y_s, were there not the hostile country. This model can be considered as a nonlinear, discretized version of Richardson's original equations [58].

Among the surprising features of the above *resource-limited* model are that in general it shows a *sharp* transition to instability, with no preceding *period-doubling* (or other familiar) scenario to signal the threatening instability, (interpreted as the outbreak of war or a break-down of the model) and that the *threshold* to instability is *much lower*, if both countries are more independent and not unrealistically assumed to react in precisely the same way.

A.3: Global Resource Limitation

A slight variation of this model with significantly different properties is given if the resource limitation affects all arms expenditures. Mathematically, one obtains:

$$\begin{aligned} x_{t+1} &= x_t + \Big[-k_{11}(x_t - x_s) + k_{12}y_t\Big](x_m - x_t) \\ y_{t+1} &= y_t + \Big[-k_{22}(y_t - y_s) + k_{21}x_t\Big](y_m - y_t) \end{aligned} \tag{12}$$

The simple *global resource-limited* model Eq. (12) seems to behave according to our expectations: if the mutual defense intensity k is small, reflecting confidence into the other country's peacefulness, there is a rather low level military expenditure, only slightly above the self-caused level s, increasing with k. If k exceeds a threshold value k_c (range $k_c \leq k < 2+k_c$) the countries spend the highest possible expenditure for arms, irrespective of their internal demands. An arms race is unavoidable if the mutual defense factor is above k_c. The change from the low-level to high-level armament occurs continuously with k, i.e., more or less undetected by the policy makers or public.

A.4: Strategic Defense Initiative (SDI) Model

A more specific arms race model, based on the same mathematical tools of discrete-time dynamics, has been applied to model the impact of strategic defensive systems (SDI) on the superpower arms race [61]. This simple model only includes three S.D.I. elements: (a) intercontinental ballistic missiles (ICBMs), the component of offensive, strategic, nuclear warfare; (b) anti-ICBM satellites, designated to attack and destroy ICBMs from space; and (c) anti-satellite missiles or other weapons launched to destroy overhead anti-ICBM satellites before these satellites can destroy the ICBMs.

The results of this model caricature indicate that for most parameter combinations, the introduction of SDI systems leads to an extension of the offensive arms race rather than to a transition to a defense-dominated strategic configuration. A reduction in the number of offensive weapons, i.e., an approach to a defense-dominated strategy, was observed if either the number of reentry vehicles per ICBM (MIRV) is limited to much smaller values than presently realized or if the accuracy of offensive weapons is significantly reduced. For the case of a strongly accelerated arms build up (either offensive or defensive) one observes a loss of stability of the solutions that represents a transition to chaotic behavior of the equations and is interpreted as unpredictable political consequences of a given action by one side.

In practice, this model was studied [61] computationally by iterating the model equations including the influence of *stochastic perturbations* (applied at every time step) over a given period of time and then by determining the range of observed outcomes from the simulation. The stochastic nature of these perturbations is very important for reliable robustness estimates: in contrast to linear systems, in nonlinear systems the responses to separate perturbations to individual variables or parameters can not be used to draw inferences on the system's behavior under simultaneous perturbations of many different factors. Somewhat facetiously we observe that Murphy's law pays respect to this phenomenon: it often seems that many disadvantageous factors in a complex system *conspire* to achieve the worst possible outcome.

B. A Three-Nation Richardson Model

In this subsection we turn to an extension of the Richardson equations to model an arms race among three nations; this introduces the additional complication of possible alliance formations (see for example [74]). With three nations the natural choice for an alliance is between the two weaker nations, as otherwise the two nations in the alliance would have dominant superiority over the third one, and the competition could be reduced to that between the nations within the alliance. For the model, let us therefore assume that we have a situation in which the nations X, Y, Z have normalized arms expenditures amounting to values x_n, y_n, z_n, where the index n indicates a discrete time unit and corresponds to a typical decision period : for example, x_n can indicate the military budget in year n. We study the generic situation in which $x_n > y_n > z_n$. In this case, by our assumptions Y and Z would form an alliance against X. The equations describing the armament level in the year $n+1$ contain, for each nation, four relevant factors: (1) the armament level in the previous year (x_n, y_n, z_n); (2) the intrinsic self-armament level that each nation wants to acquire, independent of external threats (x_s, y_s, z_s); (3) the external threat from hostile nations; and (4) the economical limitations $(x_{max}, y_{max}, z_{max})$. The terms in the equations describing the external threats are different for nations which have to defend themselves against an alliance than for those who are in an alliance. In the first case the threat consists of the sum of the expenditures of the adversaries, $(y_n + z_n)$, whereas in the second case the threat from the adversary X is reduced by the expenditures of the ally $(x_n - z_n)$ for Y, $(x_n - y_n)$ for Z.

We regard the arms expenditures – the x_n, y_n, z_n – as the variables whose time evolution we want to follow and the other variables – x_s, y_s, z_s and $x_{max}, y_{max}, z_{max}$ – as parameters, as in the resource-limited models discussed above. Here there are several parameters, which effectively represent the results of political decisions about rates at which goals are to be approached. The parameters (k_{11}, k_{22}, k_{33}) determine how fast each nation tries to achieve its intrinsic armament level independent of external threats, while the parameters (k_{23}, k_{13}, k_{12}) describe the rate at which a perceived external threat is countered. For the condition that $x_n > y_n > z_n$, the resulting discrete are given by:

$$x_{n+1} = x_n + \Big[k_{11}(x_s - x_n) + k_{23}(y_n + z_n)\Big](x_{max} - x_n)$$

$$
\begin{aligned}
y_{n+1} &= y_n + \Big[k_{22}(y_s - y_n) + k_{13}(x_n - z_n)\Big](y_{max} - y_n) \\
z_{n+1} &= z_n + \Big[k_{33}(z_s - z_n) + k_{12}(x_n - y_n)\Big](z_{max} - z_n)
\end{aligned} \tag{13}
$$

If in the course of the simulation, the ordering in the arms expenditures is changed, the equations are rearranged in such a way that a new alliance is formed among the new minor powers. In view both of the large number of parameters and of the uncertainty in their individual values, one should study [49] the equations for a wide range of parameters and attempt to isolate those parameter regions for which the solutions show robust, regular behavior from those in which sensitive, chaotic behavior is obtained.

A preliminary analysis [49] of typical solutions of this model shows (unsurprisingly) multiple attractors, separated by surfaces forming boundaries of the basins of attraction. There are indications of *small scale* chaos in several regions of state space for certain combinations of parameters. It appears also that near the boundaries in parameter space at which transitions in the alliance structures take place, the system becomes (again, unsurprisingly) very sensitive to external noise. This phenomenon could be helpful in the computational identification of crisis domains.

Although no one could take the conclusions drawn from any of the over-simplified models discussed in this Section as the central basis for any political decision, we feel that the attempt to understand and to quantify various causal relationships will lead to increasingly sophisticated models for specific aspects of these problems; when such models are produced, one will certainly need the insight gained from simpler models and the techniques of modern nonlinear dynamical systems theory to analyze these models.

5 IV. Future Outlook

A. Using Machine Learning to Find Optimal Solutions

Human intuition and pattern recognition capabilities are very impressive in several different areas of problem solving. We can all develop skills which allow us to solve recurrent problems (like playing video games) with amazing efficiency. We are also able to combine efficient solutions to seemingly unrelated problems and to synthesize them to find new solutions to more complex problems. This is probably the manner by which most progress in science and technology is made. Finally, the human brain can come up with completely new and creative solutions, which have practically nothing to with previous experiences and earlier solutions. These are exceptional cases which typically lead to break-through discoveries (while riding in the subway or looking at clouds)[3]. Limitations become evident, however, when the factors involved in considering a solution to a give problem exceed a certain critical complexity. Then we often regress to personal experiences, folk wisdom, astrology etc. On the other hand there exists a long tradition in creating *problem-solving*

[3]There is some evidence that certain chaotic dynamics features are used by the brain itself in order to make best ***sense of the world*** (see e.g. [25], [54])

tools which primarily enhance our information storage capacity [4]. These *tools* range from such *simple* constructs as nursery rhymes and songs (utilizing the associative capacities of our memory) through rudimentary computational tools like pencil and paper to advanced computer simulation and visualization environments, which permit one to simulate and observe in rapid succession many different *realities* . Indeed, today there are methods available which simulate on computers some aspects of at least three categories of human problem solving. A very widely used method in Artificial Intelligence (AI) borrows explicitly from what we know about brain functions related to human learning: Neural nets have found applications in many diverse fields including some of the AI features of the smart weapons used in the gulf war (see [23] for a recent overview). We have used another approach, based on genetic algorithms, for searching in a 15-dimensional space of parameters and initial conditions of the generalized Richardson equation [26] for solutions which minimize imbalances in armament levels.

In such cases, the parameter space is so large that it is infeasible to search it exhaustively. Since each equation must be iterated until it reaches asymptotic behavior for many different initial conditions and tested for sensitivity (by seeing whether small changes of initial conditions lead to qualitatively different asymptotic behavior), each point in the parameter space is expensive to evaluate. Genetic algorithms offer the possibility of searching the parameter space intelligently to find regions of interest. As an estimate, we iterate the model for twenty time steps and compare the spending of X (the dominant country) with that of Y and Z (the allied countries). The imbalance function that we want to minimize is given by

$$F = |x_{20} - (y_{20} + z_{20})|.$$

In the model there are twelve independent parameters (fifteen if the initial conditions are included). These real-valued parameters are defined to be in the range $(0, 1]$, so it is straightforward to discretize them into bins. This results in one binary-coded integer for each parameter designating its bin. The size of the bin is determined by the number of bits used in the discretization (for example 8 bits corresponding to $2^8 = 256$ different bins for each parameter were used in [26]). The genetic algorithm finds balance-of-power solutions which were not known to exist previously. Manually setting the parameters had not suggested that such points existed. These results are non-trivial in that the genetic algorithm found many different solutions and most of the solutions were not degenerate (that is, parameters were not set to 0.0 or 1.0). This is in contrast to a similar search using neural networks, which has produced results which often suggest trivial solutions. Studying hyper-planes of parameter space shows that it contains regular subregions with close to perfect fitness. That suggests for our model that once we have found one solution with the genetic algorithm, it is easy to find other solutions in a neighborhood due to the smoothness of the system. An illustration of this phenomenon is demonstrated in Fig. 7

[4]There is an empirical rule that we can only spontaneously grasp and remember information which does not exceed the equivalent of 7 decimal digits, like in US phone numbers.

B. Forecasting and Control of Chaos

Much progress has been made recently in the problem of reconstructing non-linear dynamical equations from observed data. (see e.g. [14], [21], [16], [22]). The hypothesis is that these equations would be equivalent to the original dynamical system which has produced the observed data. The reconstructed nonlinear model then can be used to forecast future values of the observed variables. Results from simulated and also experimental data from physical measurements indicate that even in relatively complex systems some deterministic element of the dynamics can be detected by these methods and thereby yield greatly enhanced accuracy for forecasts. The direct application of these methods to political systems and arms-race problems is obstructed by the fact that the available historical data are very sparse and that the system is exceedingly complex and undergoes perpetual evolution [11]. The other extreme to global models with very long time scales are those which deal with large amounts of data over very short time spans. For example opinion polls seem to be used frequently to estimate the response of voters to decisions by the government. It is quite feasible that those short lived variables can enter statistical models which then will influence, e.g. arms control negotiations. We have to expect that progress in nonlinear dynamics and chaos will be applied to utilize the deterministic features of these multi component data for improved forecasting (see e.g. [53]). Once we have a good model of the system we are interested in, it is natural that we can use our knowledge to control the system to perform in a specific way: since chaotic systems are so sensitive to small perturbations, it also should be possible to use this enormous amplification of small perturbations to influence the system in a desired way. Instead of controlling a chaotic system through fast feedback mechanism –which under certain conditions can actual exacerbate the situation by themselves becoming the origin of new chaotic behavior – one can use the intrinsic *attracting* nature of nonlinear systems with dissipation *entrain* the behavior to a desired dynamics ([34], [20],[50]) . This is a generalization to chaotic solutions of an old technique widely used in the context of periodic signals to tune FM receivers.

For a series of model examples and even for some actual experiments it has been shown that an optimal chaotic control exists [34]. It minimizes the energy necessary for the control and again can be many orders of magnitude more efficient than more traditional approaches. This new technique is especially useful for cases in which feedback is not possible, for example, when the dynamics consists of many independent or weakly-coupled units which do not behave coherently. The difference between this type of control and feedback control can be illustrated in an example: Assume a government wants to introduce a new policy, on drugs, say. The analog of feedback control would be to observe the system (identify drug users) and force them back to the desired mode of behavior (punishment, rehabilitation). In the open loop case the policy decisions would be independent on the state of the current drug user population. The government would generate public campaigns, education, role models which encourage the desired behavior in general, independent of the individual's current behavior. As one can imagine, such an open loop control cannot work statically,it would require very creative, ever changing efforts to attract the attention of the target population. It appears that in many chaotic systems there exist regions of stability where external perturbations are rapidly extinguished. If one imposes an external control force (of small magnitude) onto the system in an area of stability then this effect can be used to "entrain" the originally chaotic system in a new state of almost arbitrary behavior [36]. The connections to propaganda and mass control in political systems is apparent and we want to mention the potentials but also the dangers of these methods in combination with powerful computer and communications networks.

It is also not hard to speculate on the utility of such a chaotic control methodology in environmental problems. For instance, in the atmosphere we have a complex situation with an inhomogeneous distribution of temperature, pressure, and moisture. Under certain conditions self-organization of these variables occurs and hurricanes or tornados are formed. Understanding the dynamics of those modes of behavior might make it possible to control the creation and destruction of such coherent structures through using the chaotic control methods of nonlinear dynamics.

On a still more speculative level, in studying the global climatic problems such as the greenhouse effect and ozone depletion we may learn that it is already too late to reverse the negative effects of man-made perturbations, such as carbon dioxide emissions, simply by reducing these perturbations. Perhaps we will be compelled to take active steps – for instance, creating hurricanes on the ocean to enhance carbon dioxide exchange, preventing the hurricanes from touching land, steering rain clouds to areas where desertification is threatened – to reverse these trends; a workable chaotic control theory would clearly be valuable in this context. Similarly, if we can build models which predict when and where earthquakes are likely to happen, we might trigger the earthquake through well-calculated small shocks and thereby make sure that it doesn't come as a surprise and that we have enough time to evacuate threatened areas. Again, chaotic prediction algorithms may prove relevant.

Adaptive and Evolving Systems

In the previous discussions on global models of arms races the dynamical laws had to be incorporated explicitly into the model. For example in the 3-Nation Richardson model there is a specific condition in the equations which determines the nations who form alliances against whom. An alternative approach - which is closely related to the one discussed under machine learning - tries to arrives at those global conditions as *emergent properties* of the self-organized structures arising from the individual subsystems. That means in this example that there was no condition in the original equations which related to the formation of alliances. This global condition should arrive as a natural consequence of the individual's pursuit of his/her own *happiness* (see e.g. [4]). In a very general context the theory and simulations of this emergent cooperative behavior is discussed in the new area of the study of artificial life [37]. One of the general results of these studies seems to be the realization that interesting and non-trivial evolution appears *at the edge of chaos* , were conditions are neither too restrictive and suppressive but also not completely unbounded and structureless.

New Computational Tools, Scientific Visualization/Audification, Interactive Games, Virtual Reality

One specific application of the concepts discussed in the previous section has been in new sophisticated computer games which can be seen as unrealistically simple simulations with entertainment goals. Specifically Maxis Software's *SimCity* and *SimEarth* (based on Lovelock's Gaia theory) utilize the computational properties of cellular automata to create whole populations of individuals in an artificial environment, who respond directly to their living conditions. The emergent behavior of the population (expressed in polls, tax income, population shifts, etc) is surprisingly complex and captures essential features of our own experience. [5]. Together with modern multi media display techniques, these simulations can create a *virtual reality* in which it will be possible to walk through scenarios of political decision making before implementation in the real world with its irreversible consequences (see e.g. the current burning oil fields in Kuwait). Computer aided design today provides software models of new products from airplanes to buildings. The sports car manufacturer Porsche runs (and crashes!) new designs of sports cars first on the computer because real crashes would be far too expensive. Airplane pilots who have been exclusively trained on simulators are generally safer than those who had part of their training in real airplanes. Maybe similar computational simulations of varieties of political scenarios of responding to a given problem could reduce the damage done through many political decisions.

Closely connected with the development of chaos theory was the breath taking progress in the performance of computer hardware and software: Many results would have been practically impossible without the availability of fast, interactive computer graphics and scientific visualization tools. Currently there are developments are under way to utilize also audio senses to represent complex, multi dimensional data structures. Similar progress has been made in business areas where decisions are made with increasing reference to computerized data bases and electronic spreadsheets. As in many parallel developments we have to expect some convergence and synthesis: problems which require decisions with far reaching consequences need to be presented in a way that the decision maker can directly relate to them as in electronic spreadsheets. Traditionally these were static, did not include statistical analysis and dynamic simulations. Modern, spreadsheets, however, incorporate many features from simulation and visualization requirements. Thus we can formulate the Richardson arms race models in terms of a spreadsheet and thus are able to combine them in a natural way with existing methods of analysis and data handling (see e.g. Fig. 8).

More recent object-oriented computational tools like *Diagram* for the NeXT allow the interactive integration of conceptual models into network diagrams with arbitray object attached to each of the nodes of the network. Those can be plain text documents explaining the nature of the node, thay can also be sound, graphics, or video objects or, in a self-referential manner, diagrams on a lower level, indicating what factors contribute to the given node. The nodes can also represent complete simulation programs running either on the same or on a remote machine.

[5]Perhaps those computer games will one day take on the role that fairy tales and metaphors have today with their strong influence on political rhetoric and thinking

Thus we can create very complex hierarchical structures which are either dynamic or static. The dynamical models that we use to explore their interactions can be either global or explore local aspects of the whole system. This is a fundamental difference to the classical world-models, which consist of codes of tens of thousands of lines, and where, once they are written and tested, it is almost impossible to make local adjustments due to, e.g. onforeseen factors or developments, like the recent changes in Eastern Europe. Fig.9 gives a simple example of the structure of such a world diagram. In this case we have used the UN program on Human Dimensions of Global Environmental Change as a basis and also incorporated some more specific events of current interests.

V. Discussion and Conclusions: The Lessons of Nonlinearity

Despite the rather sweeping title of our article, we have in fact discussed only a limited number of the challenges that the inherent nonlinearity of environmental and socio-political issues pose for mathematical modeling. For instance, by focusing on mathematical models expressed in terms of ordinary differential and difference equations, we have tacitly omitted the significant class of systems in which spatial variability is important. An environmentally significant example of this class of problems is fluid flow through porous media, which is essential to modeling both advanced oil recovery and ground water pollution issues. These systems have typically been modeled by partial differential equations, which allow for continuous variation in time and in the spatial degrees of freedom. Recently, however, more discrete methods – loosely speaking, conceptually analogous to the use of the discrete logistic map in contrast to the continuous-time logistic dynamics – have proven to be of considerable value in analyzing such problems [32].

Nonetheless, even within our limited selection of topics we have been able to describe and illustrate many important concepts, including some that have only recently begun to creep into public awareness. First, we have learned to expect that nonlinear systems will exhibit *bifurcations*, so that small changes in parameters can lead to qualitative transitions to new types of solutions. The global greenhouse effect, discussed in this conference, may be an example of such a phenomenon: a small increase in the carbon dioxide concentration in the atmosphere may lead to a qualitative change in the global ecology. Similarly, possible bifurcation phenomena must be kept in mind in analyzing models both of environmental phenomena like ozone depletion and deforestation and of socio-political phenomena like population growth, trade imbalances, and the effects of technological innovations. Second, we have seen that apparently random behavior in some nonlinear systems can in fact be described by *deterministic chaos*, with its exquisite sensitivity to initial conditions and complex, aperiodic motion. The fundamental limitations on our ability to predict the weather are one consequence of deterministic chaos. Third, we have argued that in typical nonlinear systems there will exist *multiple basins of attraction* and that the boundaries between these different basins can have incredibly complicated *fractal* forms. This represents another form of sensitivity to initial conditions and consequent limit to prediction: even if all possible long-term behaviors of a system are simple, determining which one will occur for a given (imprecisely known) initial state can be essentially impossible.

On the surface, these three insights seem rather depressing, since they appear to provide clear examples of limitations to our potential quantitative understanding even in the cases for which we have reliable models. But at a deeper level, there are many reasons to be encouraged about the future of mathematical modeling of complex nonlinear problems. Heightened general awareness of of the subtle inter-connectivity and the limits to what we can know about the global consequences of local actions may lead more care and restraint in confronting environmental issues. This prospect is essentially a positive restatement of our earlier remark that models guide our perceptions of what can happen as well as what is likely to happen. Further, on a more technical level there are several promising developments. The *universality* of certain nonlinear phenomena implies that we may hope to understand many seeming disparate systems in terms of few simple paradigms and models. Thus, there is the possibility of what might be termed *intellectual technology transfer* , in which features observed and understood in one phenomenon can be used to interpret another. In addition, the fact that deterministic chaos does follow from a well-defined dynamics with no random influences means that in principle one *can* predict behavior for short periods of time. In the case of the weather, for example, predicting the weather ten minutes ahead – old jokes about the weather in the mountains of New Mexico aside – is something that any of us can do by looking out the window. Although this seems obvious, recall that if the weather were really random on a moment-to-moment basis, even this limited prediction would be impossible. Recently, there has been considerable progress in developing more formal approaches to both forecasting and noise reduction using the theory of deterministic chaos (see for example [22] and references therein). These approaches have led to predictions for the behavior of certain chaotic systems which predictions are orders of magnitude more accurate than those based on classical forecasting methods. A more colorful (and earlier) example is described in "The Eudaemonic Pie [5], in which, by making a series of measurements with the aid of a computer hidden in a shoe, students of chaos theory were able to *predict* outcome of a roulette wheel with sufficient accuracy to beat the system (at least as long as the computer worked !!).

After such flights of fancy, it is perhaps most appropriate to end with the essential caveat that applies to all attempts to quantify complex issues such as those facing us in the environmental and socio-political arenas. While mathematical models can force us to quantify our understanding of the causal relations among the various components of a complex issue and while the ability to analyze these models with the tools of modern nonlinear dynamics can sharpen our perceptions what may occur, the fundamental limitations caused by the very complex nature of, and the inherent uncertainties in, these problems mean that these models must always be interpreted, to paraphrase Dickens, as images of what might be, rather than what must be. They can never substitute for human analysis, collective wisdom, and political will. Nonetheless, coupled with these attributes, mathematical modeling of environmental and socio-political issues can contribute vitally to the development of a new and more stable world order.

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Figure 1: Exponential growth solution for Malthus' equation with positive growth factor. Without non-linear saturation the solution diverges to infinity.

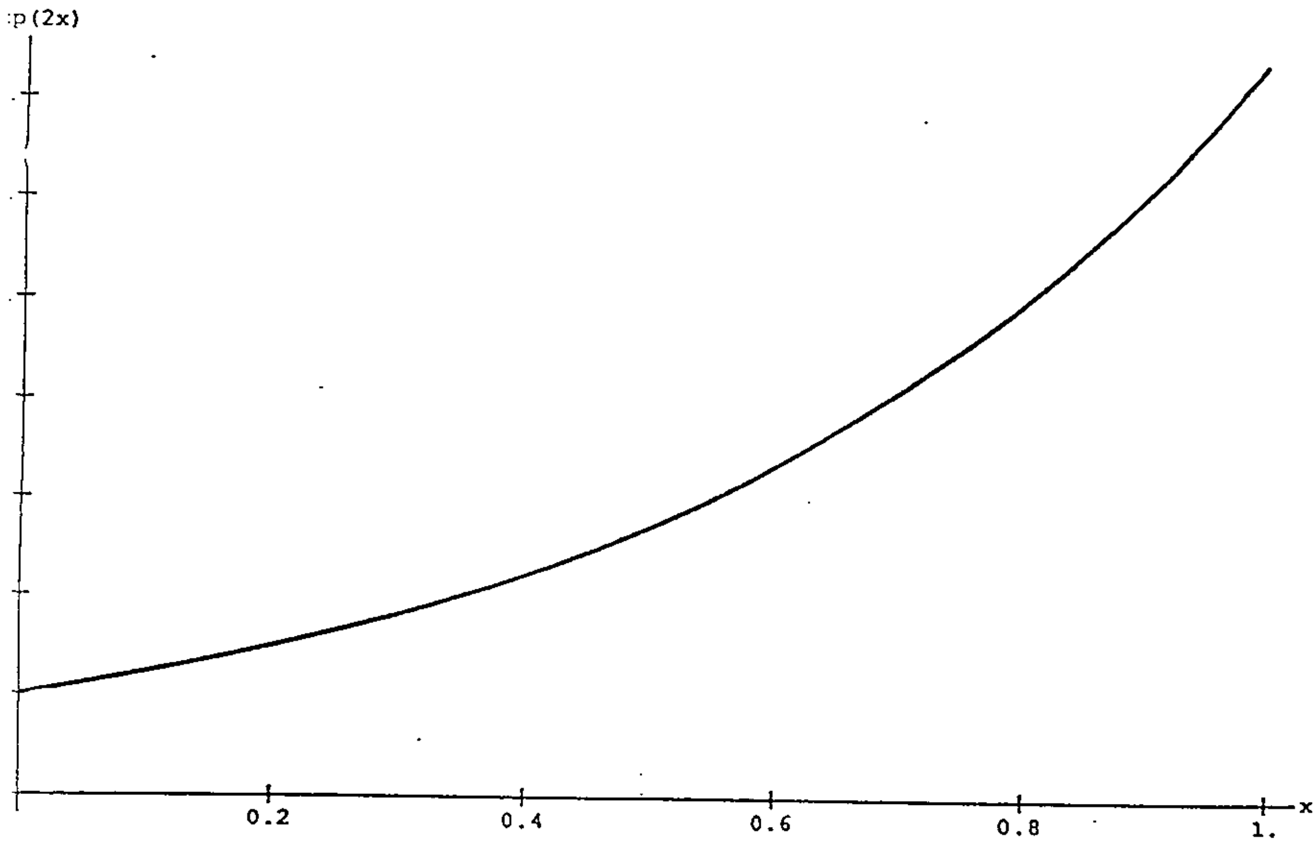
Figure 2: Exponential decay solution for Malthus' equation with negative growth factor. Asymptotically this solution tends to zero as time goes to infinity.

Figure 3: Several solutions of the Verhulst (logistic) equation for different initial conditions. All solutions with positive initial conditions approach asymptotically the stable fixed point solution $N = N_\infty$.

Figure 4: Chaotic solution of the discrete logistic equation for parameter $r = 4$ and initial condition $x_0 = .499$. The solution comes arbitrary close to every point in the interval but spends most of the time close to the endpoints of the interval. (Because of the singularities in the invariant measure.)

Figure 5: Bifurcation diagram of the discrete logistic equation for parameter $3 < r < 4$. The attractors of the system are plotted as point sets in vertical direction versus bifurcation parameter r. Bright areas indicate periodic attractors (finite point set, with period indicated by labels to arrows at top and bottom of the figure. The parameter r_c indicates the accumulation point of period-doubling bifurcations, i.e. the solution corresponds to the limit of a periodic orbit of period $P = 2^n, n \rightarrow \infty$. (Figure courtesy of J.D. Farmer, J.P. Crutchfield, B. Huberman, Physics Reports, **92**, 45, (1982))

Figure 6: Projection of the Lorenz attractor onto the $x - z$ plane. The attractor is chaotic and forms a fractal object of dimension slightly large than two ($d_F \approx 2.07$). This indicates that solutions are wandering erratically on a geometrical object which locally consists of infinitely many closely spaced surfaces.



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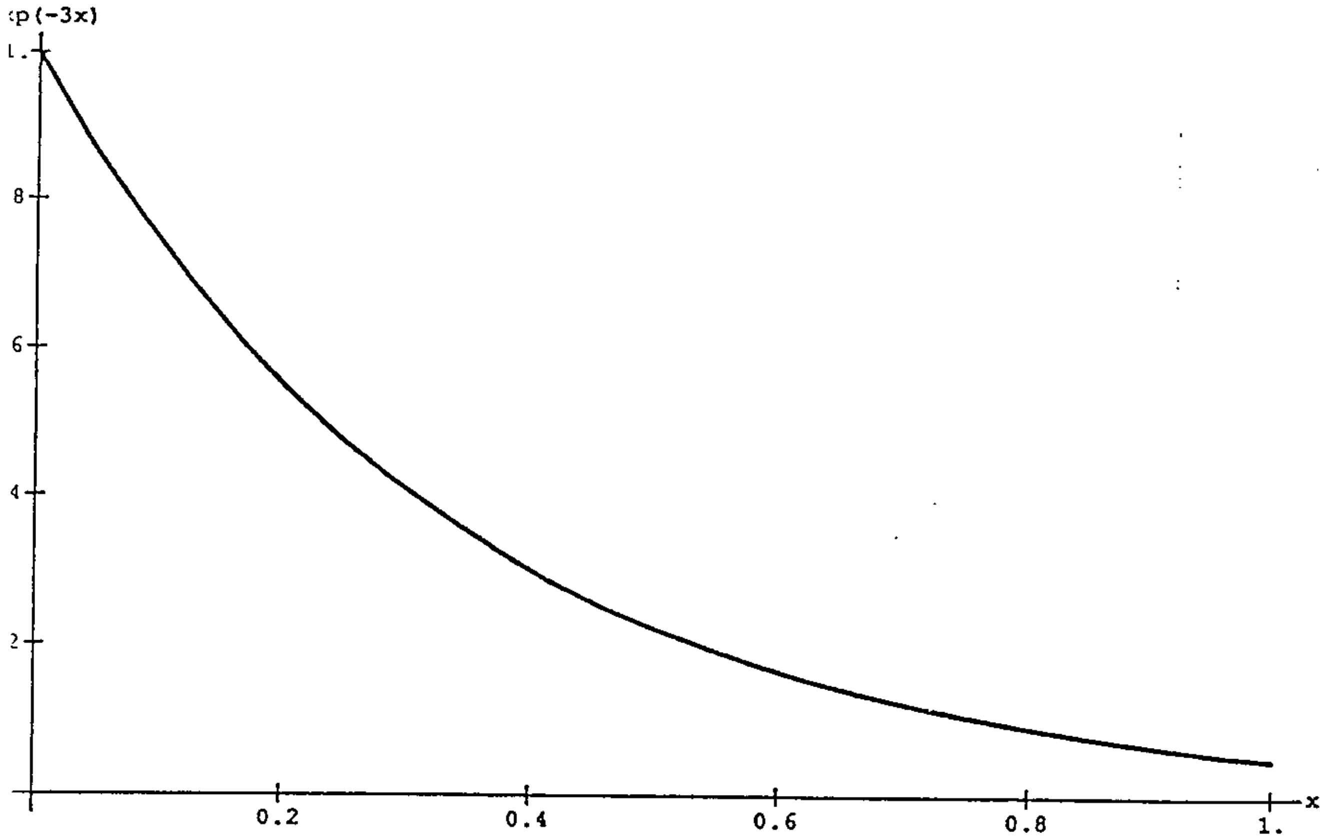

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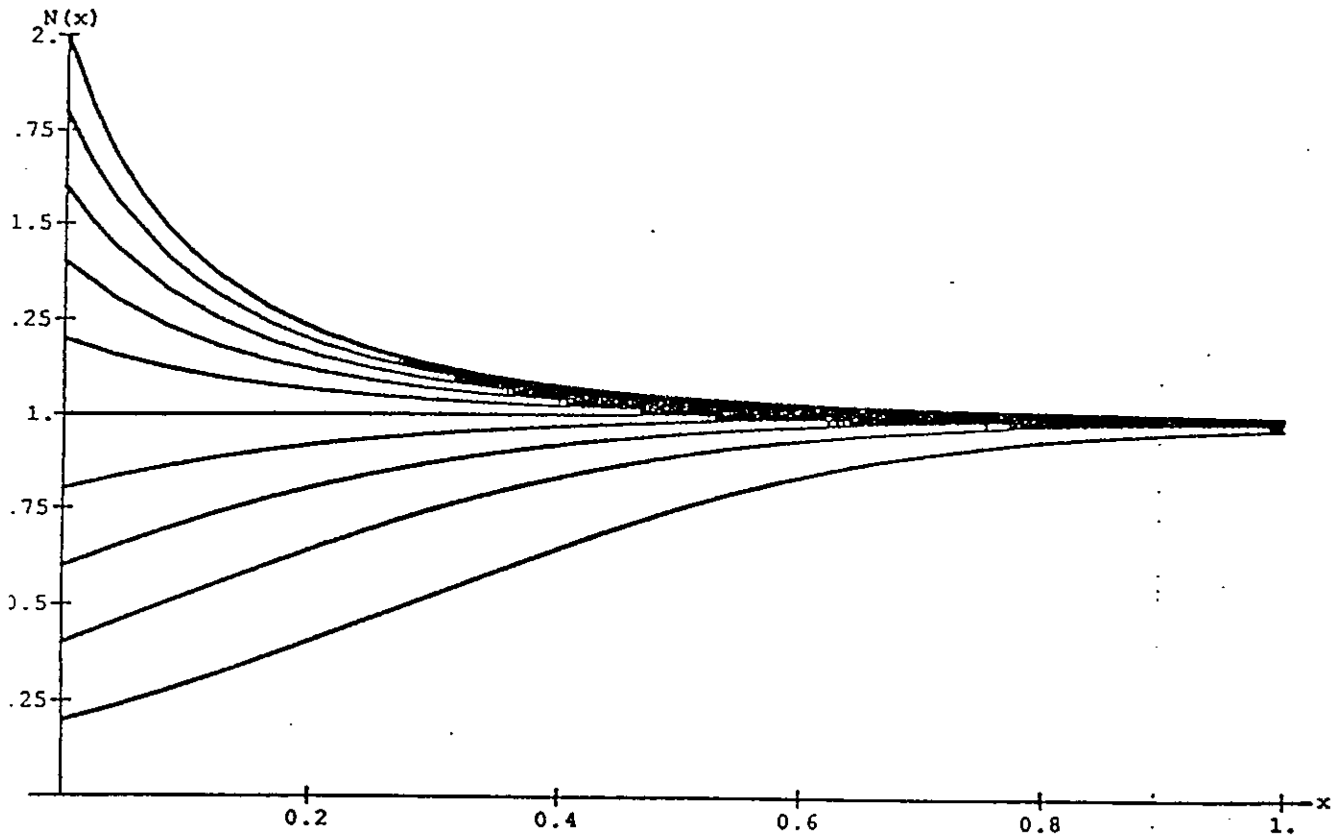

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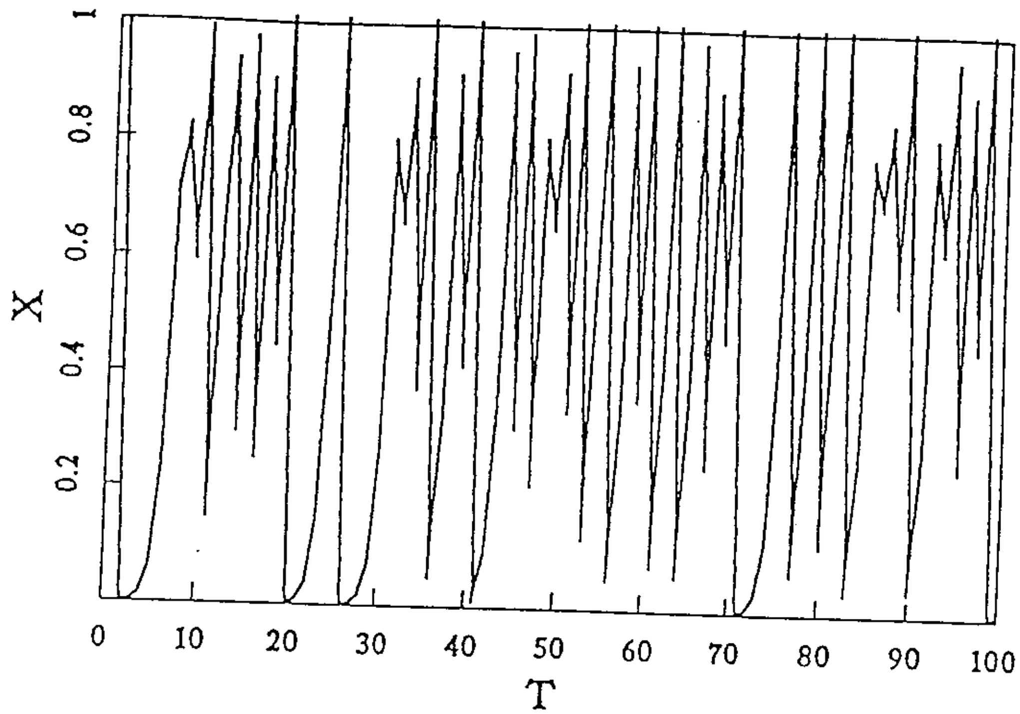

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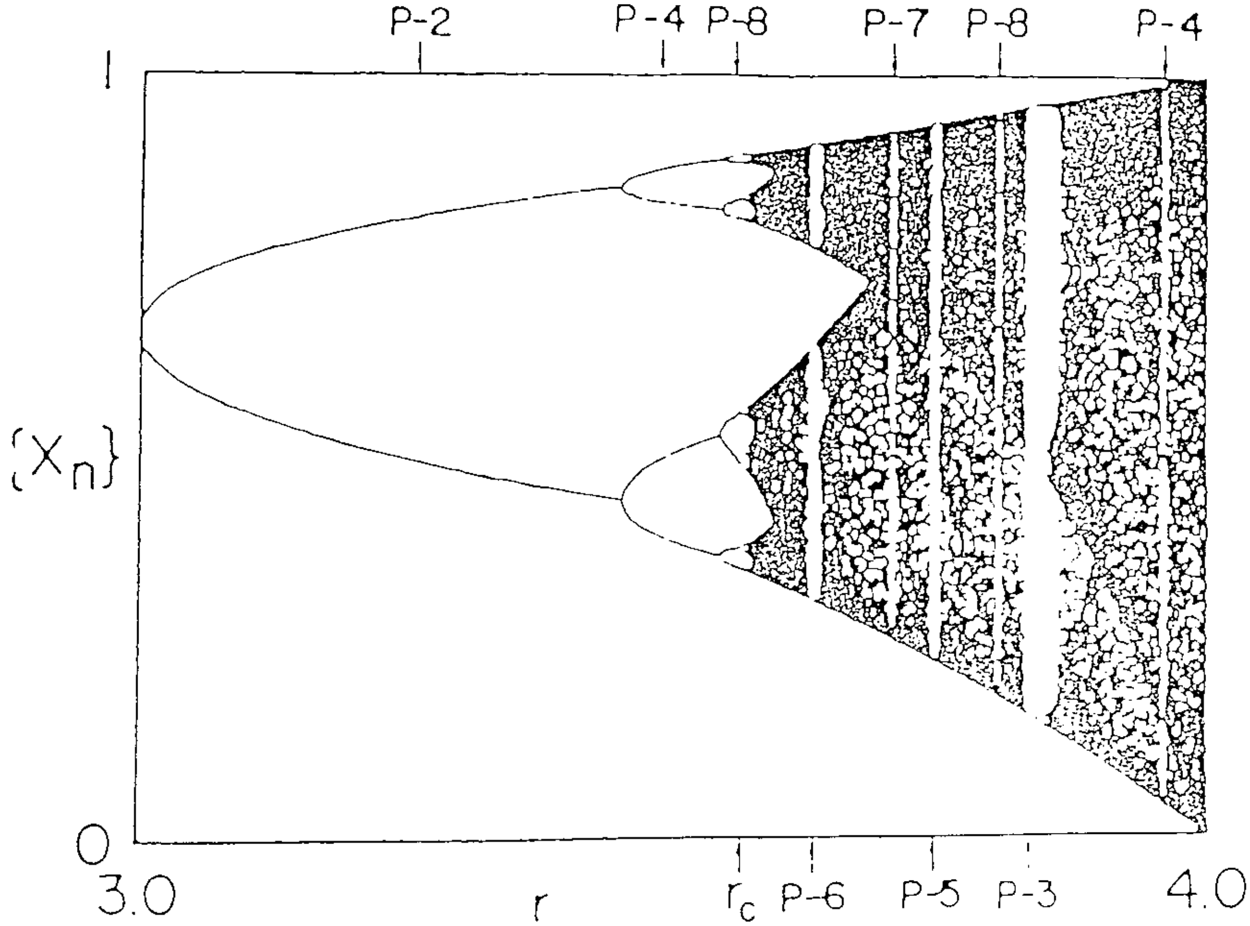

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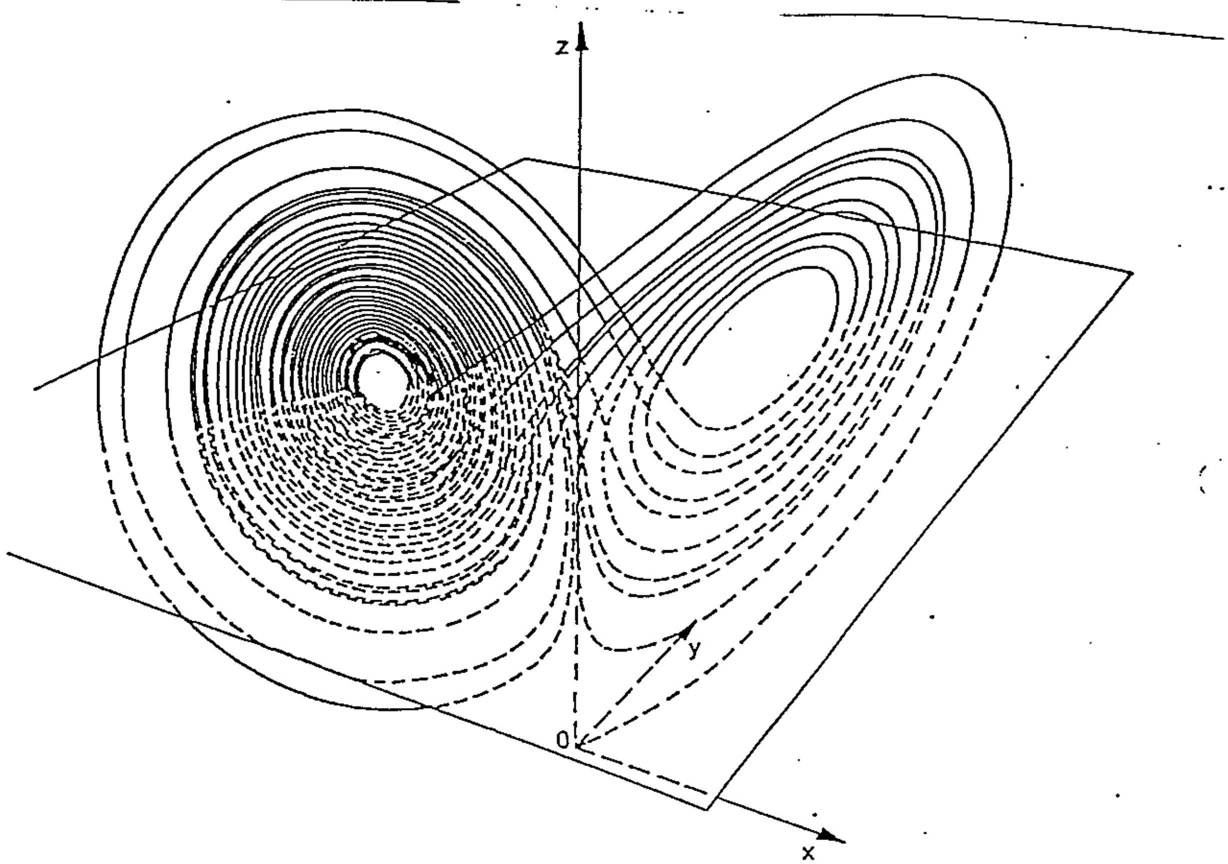

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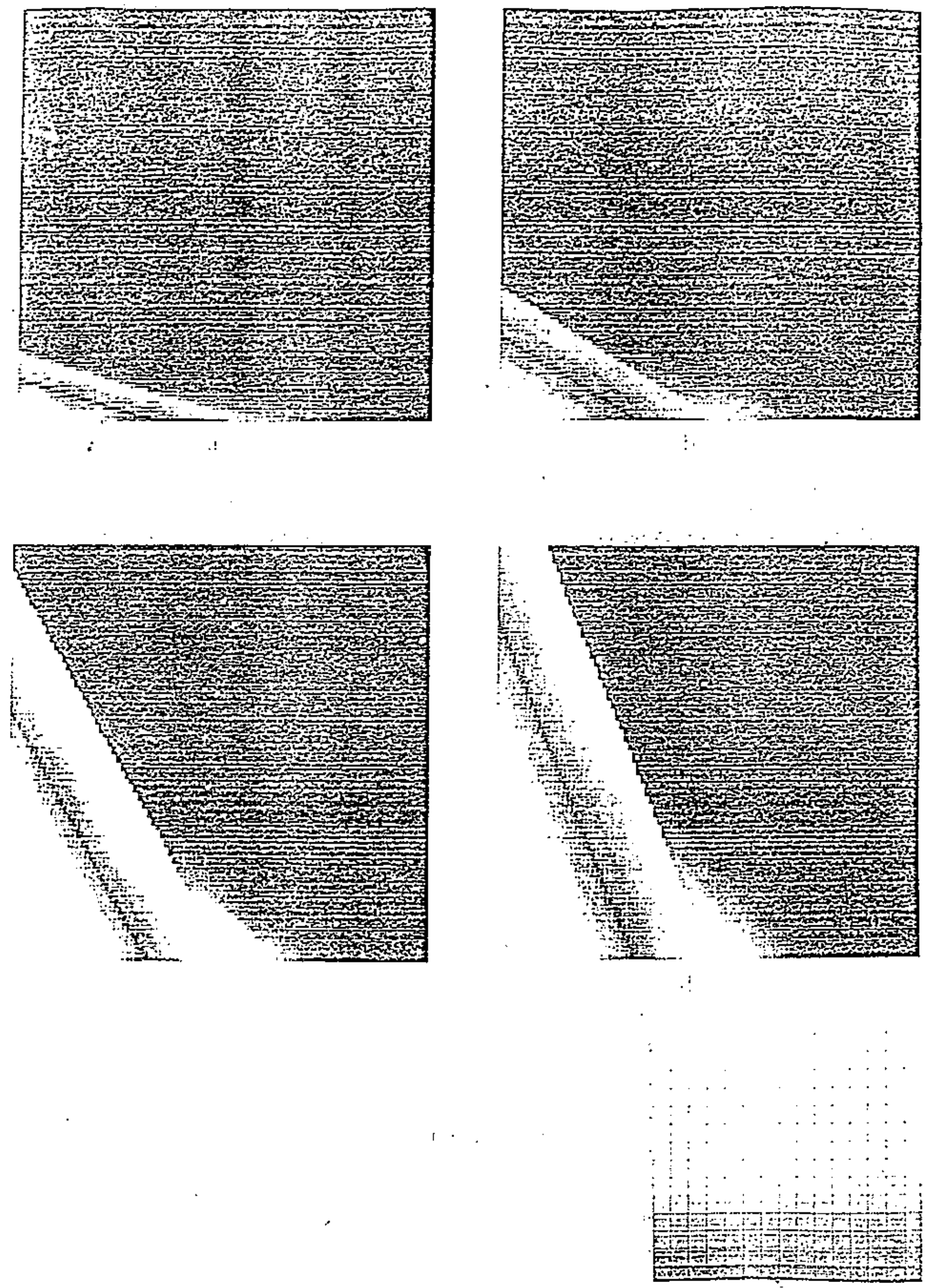

Figure 7: This figure illustrates the structure of the imbalance function which describes the distance from perfect balance of power. It is evaluated for a 2-dimensional hyperplane through one of the parameter points, found by the genetic algorithm (GA), for which the imbalance function is exactly zero. That means 12 parameters are kept fixed at the GA values but parameters x_s, z_s are varied in the interval [0,1] whereas the value of k_{11} is kept fixed at a) $k_{11} = 0.1$ b) $k_{11} = 0.2$ c) $k_{11} = 0.6$ d) $k_{11} = 0.8280730$
The axes represent the unit interval [0,1] (evaluated at a resolution of $\delta x = \delta y = 0.01$. The distance from perfect balance of power at each point in the plain is indicated by its grey value: black = imbalance function is zero. (See grey chart.)

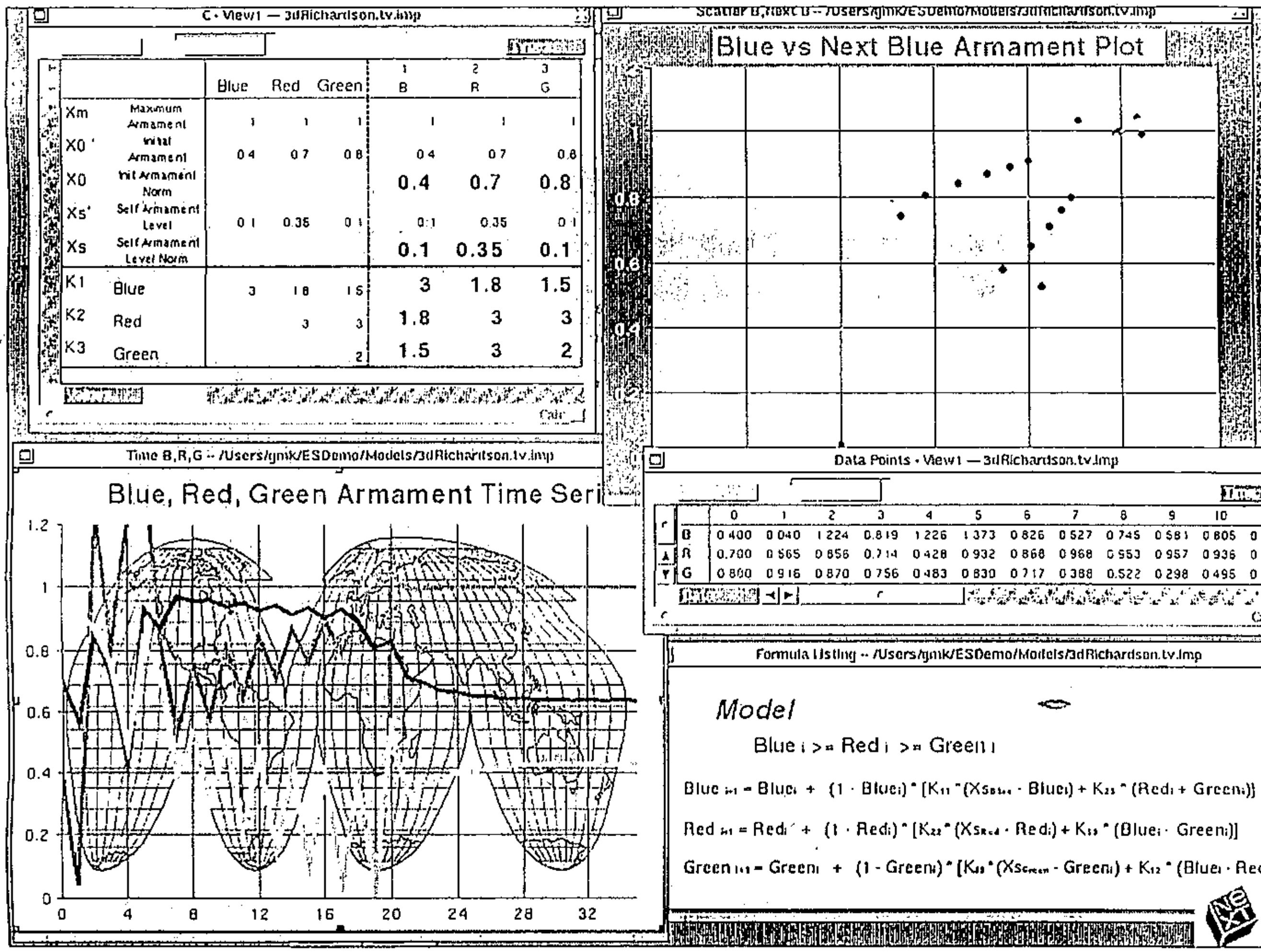

Figure 8: Screen shot of a simulation and visualization of the discrete 3-nation Richardson models discussed in the text. Different windows contain the equations (the lips indicate an icon which will narrate some descriptions of the equations), numerical timeseries, *next amplitude* plots, graphical timeseries, and a window for entering new parameters.

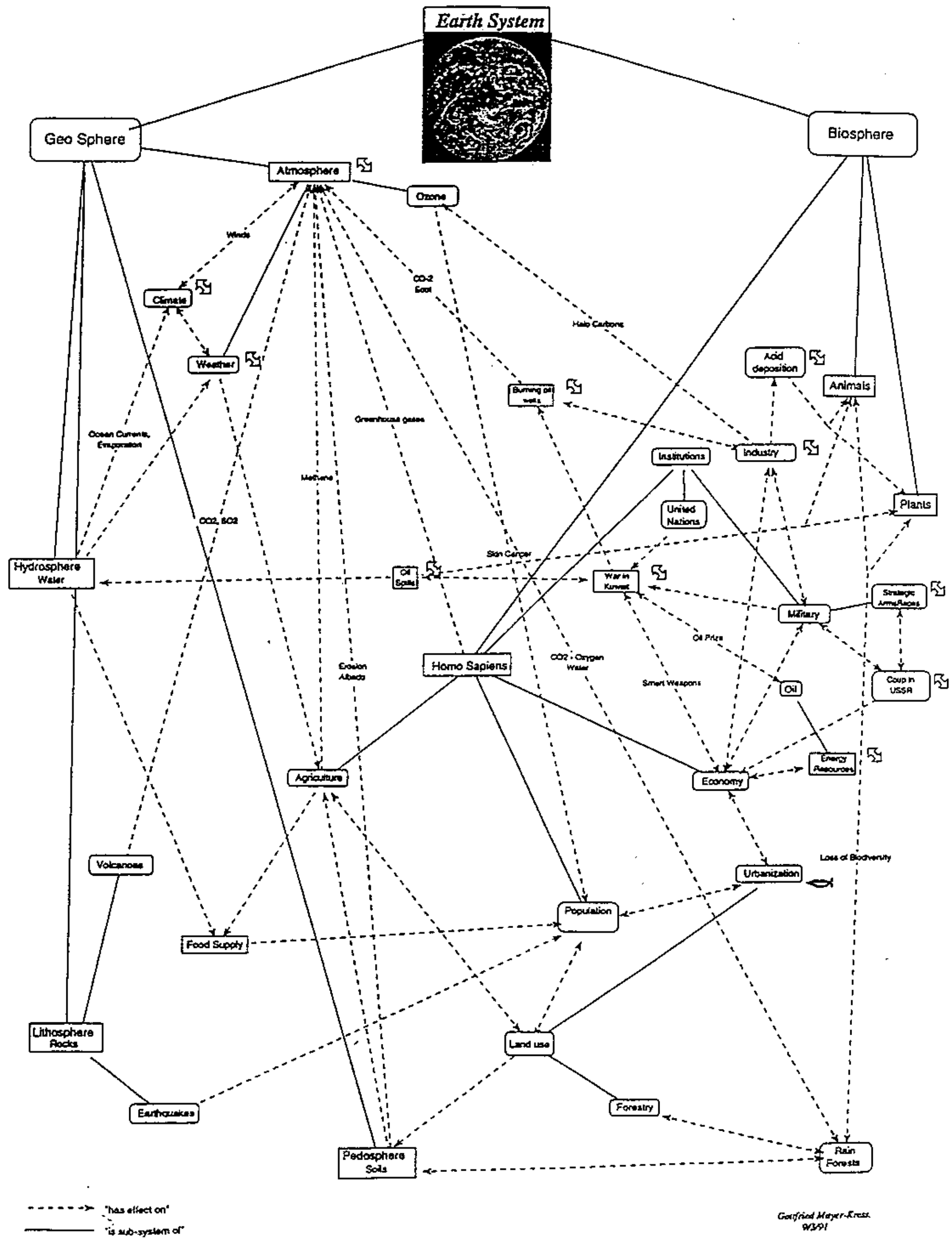

Figure 9: Simple diagram of global change related systems. This is only the top layer of a structure which represents more than 100MB of data and simulation models. Each node with an attached double arrow indicates an access to a lower level structure or object.