The Value of Government and the Efficiency of Noncooperative Equilibrium

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Abstract

Taking the maximum joint payoff of a 2×2 matrix game as a measure of potential social welfare, one can compute a simple "value of government" based upon the difference between this maximum payoff and the joint payoff obtained in noncooperative equilibrium. We construct an efficiency loss index (ELI) as the expected value of this difference divided by the maximum joint payoff, and use the ELI to analyze the amount players would be willing to pay government (or some other third-party referee) to coordinate the outcome of the game either by changing its structure or by providing signals/contracts to coordinate behavior. This analysis is applied to random games with both known and unknown opponent payoffs. We also discuss problems associated with index construction and other modeling limitations.

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1 Introduction

The strategic interaction between a pair of autonomous decision makers is of fundamental importance in social psychology, economics, political science, and biology. Such binary interactions therefore merit special attention in the theory of games.

Several cases must be considered when studying interactions between individuals: those between approximately homogeneous players who know each other's characteristics; those between players with distinctly different power levels that are known by both sides; and those with less than perfect information. Yet another set of distinctions applies to binary relationships between individuals and institutions. In some contexts, institutions simply aggregate information and match individuals (in the way stock markets pool the bids and offers of similar individuals to form prices and match traders). In other cases, however, the interaction between an individual and an institution can be highly asymmetric. Encounters between individuals and the essentially anonymous bureaucracies of government, corporations, churches, hospitals, and educational institutions are more and more frequent in a world of increasing population, and a considerable proportion of everyday life involves such interactions. Not surprisingly, technological revolution has permitted the creation of a mammoth industry to provide "personalized" window dressing for the anonymous treatment of individuals by vastly more powerful institutions.

To evaluate the anarchist's dream of fully independent, unregulated binary interactions between individuals, one must understand how the rules of such games come into being, are enforced, and possibly modified. In real life, the rules and their management impose a social cost, and a basic question to be addressed is how much individuals would be willing to pay for this governance. Confining ourselves to binary interactions that may be represented as 2×2 matrix games, we provide an answer to this question. In particular, we propose an Efficiency Loss Index (ELI),

$$ELI = \mathbb{E}\left[\frac{\text{Max. Joint Payoff} - \text{Joint Payoff from Noncooperative Equil.}}{\text{Max. Joint Payoff}}\right]$$

as a measure of the potential value of government, where "noncooperative equilibrium" refers to the unique or Pareto-dominant pure-strategy noncooperative equilibrium (NCE) if it exists, and defaults to the mixed-strategy equilibrium otherwise.[1]

[1]As will be shown below, the default mechanism is rarely employed (e.g., in only slightly

In applying mathematical models to problems in the behavioral sciences, quantitative differences often generate qualitative differences. Thus, the social interaction expected in a game with numerous individuals may be quite different from that in binary or ternary analogues. Similarly, expanding the number of strategic options per player from two or three to “many” can yield significant behavioral differences. In the present work, we offer some brief comments on the relevance of our work to various behavioral sciences, but leave for separate discussion a detailed analysis of the linking of the formal models to applications.

2 Efficiency Loss Index for 2×2 Games

2.1 Preliminary Comments

A social science index, whether for stocks, bonds, cost of living, quality of life, or weather, usually affords only a gross simplification of aggregate information. Nevertheless, such indices can be highly instructive if used with care.

The development of indices is part of an evolving process in measurement. Useful indices are often fairly crude at inception and improved over time. Ideally, it may be desirable to justify an index axiomatically, but this level of precision is often difficult to achieve. After providing a formal definition of the Efficiency Loss Index (ELI), we will note several problems that, although not fully addressed in the present article, must eventually be considered in the development of a matrix-game inefficiency index.

2.2 A Proposed Index

The 2×2 matrix game is the minimally complex, intrinsically symmetric game of strategy that can be formalized. For this reason, it plays crucial roles both in teaching elementary game theory and in social science experimentation. A generic 2×2 game is described by the 4 payoff pairs $[a_{ij}, b_{ij}]$, for $i = 1, 2$ and $j = 1, 2$, shown in Table 1.

more than 5 percent of all strictly ordinal 2×2 matrix games).

	Left	Right
Up	a_{11}, b_{11}	a_{12}, b_{12}
Down	a_{21}, b_{21}	a_{22}, b_{22}

Table 1: A Generic 2×2 Matrix Game

For simplicity, we will treat the payoffs in each cell as measurable, comparable, and transferable monetary amounts. In particular, we will assume that a_{ij} and b_{ij} can be any positive, real-valued numbers, and take the sum $a_{ij} + b_{ij}$ to be a meaningful measure of social welfare. We further will assume that the NCE or consistent-expectations model of individual behavior offers a reasonable portrait of self-interested decision making.

In computer science, there is now a small literature on the "price of anarchy" (see Halpern, 2003; Rothblum, 2006; and Roughgarden, 2009). In terms of game theory, this concept is equivalent to the loss of efficiency that arises by selecting an NCE payoff rather than the joint maximum (which presumably could be imposed by an omnipotent social planner). This concept of inefficiency has been applied to network games (see Rothblum, 2006). In the present work, we apply it to matrix games as a measure of the potential value of government.

Let $\mathcal{P}$ denote the set of pure-strategy NCE in a given 2×2 matrix game, and note that the cardinality of this set, $|\mathcal{P}|$, must equal 0, 1, or 2. We now define the following quantities:

$$MP = \max_{i,j} \{a_{ij} + b_{ij}\} \tag{1}$$

the maximum possible joint payoff; and

$$EP = \begin{cases} \sum_{i=1}^{2}\sum_{j=1}^{2} I_i I_j (a_{ij} + b_{ij}) & \text{if } |\mathcal{P}| = 0 \\ a_{ij} + b_{ij} \text{ s.t. } (i,j) \in \mathcal{P} & \text{if } |\mathcal{P}| = 1 \\ a_{i'j'} + b_{i'j'} & \text{if } |\mathcal{P}| = 2 \text{ and } \exists (i', j') \\ & \quad \text{s.t. } a_{i'j'} \geq a_{i''j''}, b_{i'j'} \geq b_{i''j''} \\ & \quad \text{for } (i', j'), (i'', j'') \in \mathcal{P} \\ \sum_{i=1}^{2}\sum_{j=1}^{2} I_i I_j (a_{ij} + b_{ij}) & \text{if } |\mathcal{P}| = 2 \text{ and } \nexists (i', j') \\ & \quad \text{s.t. } a_{i'j'} \geq a_{i''j''}, b_{i'j'} \geq b_{i''j''} \\ & \quad \text{for } (i', j'), (i'', j'') \in \mathcal{P} \end{cases} \tag{2}$$

the joint payoff from the noncooperative equilibrium (where I_i and I_j are independent Bernoulli random variables whose respective parameters, $p_i = \Pr\{\text{row player chooses Up}\}$ and $q_j = \Pr\{\text{column player chooses Left}\}$, determine the mixed-strategy equilibrium). We then define the ELI as follows:

$$\begin{aligned} ELI &= \mathbb{E}\left[\frac{MP - EP}{MP}\right] \\ &= 1 - \mathbb{E}\left[\frac{EP}{MP}\right] \end{aligned}$$

2.3 Some Examples

The ELI provides an indication of how much could be spent on changing structure and/or providing information and coordinating devices within the canonical 2×2 game to promote and/or enforce higher levels of optimality. For example, if the Prisoner's Dilemma of Table 2 is assumed a valid representation of competition within an economy, then the joint maximum is given by the payoff pair $[3, 3]$, and the single pure-strategy NCE is $[2, 2]$.

	Left	Right
Up	3,3	1,4
Down	4,1	2,2

Table 2: A Prisoner's Dilemma Game

Consequently,

$$ELI = 1 - \frac{2+2}{3+3} = \frac{1}{3}$$

indicating that fully one-third of the joint wealth attainable can be spent on modifying the game to generate an improved outcome.

A somewhat more complicated example is afforded by the Stag Hunt of Table 3. In that case, the joint maximum is given by $[4, 4]$, and there are two pure-strategy NCE: $[4, 4]$ and $[2, 2]$.

	Left	Right
Up	4,4	1,3
Down	3,1	2,2

Table 3: A Stag Hunt Game

Since the former NCE is Pareto dominant,[2] we find

$$ELI = 1 - \frac{4+4}{4+4} = 0$$

which suggests that an economy described by this game is self-regulating (see Powers and Zhan, 2008). Comparing the value of $MP = 6$ in the Prisoner's Dilemma to that of $MP = 8$ in the Stag Hunt, we note that the proposed ELI is appropriate for inter-game comparisons only in terms of relative, within-game levels of inefficiency, and not intended to assess the overall value of a game in any absolute sense.

Another game with two pure-strategy NCE is Chicken, shown in Table 4.

	Left	Right
Up	3,3	2,4
Down	4,2	1,1

Table 4: A Chicken Game

In this case, the absence of a Pareto-dominant NCE means that the players must resort to the mixed-strategy NCE given by $p_1 = 1/2$ and $q_1 = 1/2$. Hence,

$$ELI = 1 - \left[\left(\frac{3+3}{6}\right)\left(\frac{1}{4}\right) + \left(\frac{2+4}{6}\right)\left(\frac{1}{4}\right) + \left(\frac{4+2}{6}\right)\left(\frac{1}{4}\right) + \left(\frac{1+1}{6}\right)\left(\frac{1}{4}\right)\right]$$
$$= \frac{1}{6}$$

indicating a level of efficiency exactly halfway between the Prisoner's Dilemma and the Stag Hunt.

Finally, we observe that the ELI again entails a mixed-strategy NCE for certain asymmetric 2×2 games with 0 pure-strategy equilibria, such as the asymmetric Attack game in Table 5.[3]

[2] Some would argue, using a mixed strategy with appropriate probabilities, that the NCE $[2, 2]$ is risk dominant and preferred by both players. (See Harsanyi and Selten, 1988.)

[3] This example is taken from Table 15.3 of Powers (2012), where the row player is a government that must defend against a terrorist attack on either a big civilian target

	Left	Right
Up	3,2	2,3
Down	1,4	4,1

Table 5: An Asymmetric Attack Game

Here, the mixed-strategy NCE has $p_1 = 3/4$ and $q_1 = 1/2$, yielding

$$ELI = 1 - \left[\left(\frac{3+2}{5}\right)\left(\frac{3}{8}\right) + \left(\frac{2+3}{5}\right)\left(\frac{3}{8}\right) + \left(\frac{1+4}{5}\right)\left(\frac{1}{8}\right) + \left(\frac{4+1}{5}\right)\left(\frac{1}{8}\right)\right]$$
$$= 0$$

Somewhat paradoxically, a constant-sum game – also known as a game of pure opposition – is efficient because all points fall on the Pareto surface.

2.4 An Important Application

The principal motivation for constructing the *ELI* is to obtain a rough estimate of the inefficiency of NCE in contrast with fully cooperative solution concepts. Recognizing that cooperative games often require an expensive apparatus to encourage and possibly enforce cooperation, we wish to evaluate how much players of a noncooperative game would be willing to pay government (or some other third-party referee) to provide similar coordination. Since NCE are achieved in an explicitly decentralized, autonomous manner, this question is well posed both for individual 2×2 matrix games – as illustrated above – as well as for games selected at random from the entire set of 2×2 games – as discussed in the following section.

We recognize that the NCE solution concept is based upon a highly individualistic (almost autistic) view of an economy that completely ignores inter-player payoff comparisons. For this reason, it can lead on occasion to somewhat optimistic joint payoffs, especially in asymmetric games. Nevertheless, in many cases the *ELI* remains sufficiently positive to justify paying for a government or referee to change economic structure and/or provide useful information and coordination mechanisms.[4]

(Up) or a small civilian target (Down), without the choice of defending both, and the column player is a terrorist that must decide whether to attack the big target (Left) or the small target (Right). Although this particular game possesses the constant-sum property (because the entries in each cell add up to 5), this property is not a general characteristic of terrorist-attack models.

[4]Since the economy functions within a polity, and the polity within a society, one can

3 Random Games

In the examples of Tables 2 through 5, we employed only strictly ordered payoffs (i.e., the numbers 1, 2, 3, 4) for both players. This was done for simplicity, since the a_{ij} and b_{ij} may take arbitrary positive real values with ties permitted. In extending the ELI to games with random payoffs, we first study the distribution of the ELI within the set of all possible strictly ordinal 2×2 matrix games, where each such game is given equal probability. We then present a natural generalization of this scheme that permits arbitrary positive, real-valued payoffs, and precludes the possibility of ties. Although beyond the scope of the present work, the possibility of ties is relevant for modeling certain economies, and therefore identified in Section 4 as an important direction for further study.

3.1 Strictly Ordinal Payoffs

The universe of all strictly ordinal games is easily denumerated by noting that each of the payoff vectors $[a_{11}, a_{12}, a_{21}, a_{22}]$ and $[b_{11}, b_{12}, b_{21}, b_{22}]$ must be permutations of the vector $[1, 2, 3, 4]$, yielding a total of $4! \times 4! = 576$ different outcomes. This number can be divided by 2 to remove duplications arising from interchanging rows, and by another 2 to account for interchanging columns, leaving the canonical set of 144 strategically distinct games (see, e.g., Baranyi, Lee, and Shubik, 1992).

We will denote this canonical set by $\mathcal{G}$, and its individual elements (games) by $G(k) \in \mathcal{G}$, for $k = 1, 2, \ldots, 144$, with payoffs $\left[a_{11}^{(k)}, a_{12}^{(k)}, a_{21}^{(k)}, a_{22}^{(k)}\right]$ and $\left[b_{11}^{(k)}, b_{12}^{(k)}, b_{21}^{(k)}, b_{22}^{(k)}\right]$. Various taxonomies have been proposed by game theorists and experimental gamers for these 144 games (as well as for the canonical set of 726 games with ties). These include the work of Rapaport and Guyer (1966), Rapaport, Guyer, and Gordon (1966), Borm (1987), Kilgore and Fraser (1988), and Robinson and Goforth (2005). One crude natural classification is to divide $\mathcal{G}$ into games of pure opposition (with constant-sum payoffs), mixed motives, and structural cooperation.[5] In the present

view the process of selecting the polity that regulates the economy's payoffs as a meta-game, to be played by members of society (who also may be decision makers within the economy).

[5]If the number of strategies for each player, s, grows much larger than 2, it is easy to show that the great majority of games are characterized by mixed motives, whereas the

work, we sort the games by number of pure-strategy equilibria, as well as their symmetry and constant-sum properties.

Now assume that a game is to be selected at random from $\mathcal{G}$, and that each $G(k)$ has equal probability 1/144 of being chosen. Letting $ELI(k)$ denote the random ELI associated with the random $G(k)$, we further define, for any nonempty subset $\mathcal{H} \subseteq \mathcal{G}$, the conditional random variable

$$ELI_{\mathcal{H}}(k) = ELI(k) \cdot I_{\mathcal{H}}(k)$$

where $I_{\mathcal{H}}(k)$ equals 1 if $G(k) \in \mathcal{H}$, and is undefined otherwise.

Table 6 presents expected values of $ELI_{\mathcal{H}}(k)$ for various subsets $\mathcal{H} \subseteq \mathcal{G}$. From the lower-right corner, we see that the overall inefficiency of NCE in $\mathcal{G}$ is given by

$$\mathbb{E}[ELI(k)] \approx 0.0653$$

which seems fairly low. However, there are some subdivisions that yield clearly higher average values. Looking down the rightmost ("Total") column, one can see that this is true of two NCE categories: those with (a) 0 pure-strategy NCE, for which $ELI_{\mathcal{H}}(k) \approx 0.1878$, and (b) 2 pure-strategy NCE with 0 Pareto-dominant equilibria, for which $ELI_{\mathcal{H}}(k) \approx 0.2725$. This offers prima facie evidence that mixed-strategy equilibria tend to be less efficient than pure-strategy equilibria. Looking across the various rows, one sees further that both the symmetry/asymmetry and constant-sum/non-constant-sum distinctions appear to have little effect on the inefficiency of NCE.

proportion of games of pure opposition vanishes quickly, and that of structural cooperation decreases at a somewhat slower rate.

	Symmetric	Asymmetric	Constant Sum	Non-Const. Sum	Total
$\lvert\mathcal{P}\rvert = 0$	N.A. (0)	0.1878 (18)	0.0000 (2)	0.2113 (16)	0.1878 (18)
$\lvert\mathcal{P}\rvert = 1$	0.0556 (6)	0.0317 (102)	0.0000 (4)	0.0343 (104)	0.0331 (108)
$\lvert\mathcal{P}\rvert = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\lvert\mathcal{P}\rvert = 2$, 0 Pareto Domin.	0.2460 (3)	0.2857 (6)	N.A. (0)	0.2725 (9)	0.2725 (9)
Total	0.0893 (12)	0.0631 (132)	0.0000 (6)	0.0682 (138)	0.0653 (144)

Table 6: Expected Values of $ELI_{\mathcal{H}}(k)$ for Various Subsets $\mathcal{H} \subseteq \mathcal{G}$
(Parentheses contain number of games in each subset)

Although games with strictly ordinal payoffs afford certain insights, they are inappropriate for studying inefficiency in economies whose player payoffs are not drawn (with replacement) from a domain of four equally spaced units. This is not because fractional payoff values introduce distinctly different NCE. (In fact, as will be seen below, any 2×2 matrix game with real-valued payoffs is uniquely associated with one of the canonical games in $\mathcal{G}$.) Rather, it is because the possibility of fractional payoffs can cause the ELI of a given game to vary considerably for a fixed NCE structure. For example, if one were to replace the NCE payoff pair $[2, 2]$ with values $[2.9, 2.9]$ in the Prisoner's Dilemma of Table 2, then the ELI would be lowered to

$$ELI = 1 - \frac{2.9 + 2.9}{3 + 3} \approx 0.0333$$

On the other hand, if the joint-maximum payoff pair $[3, 3]$ were replaced with $[3.9, 3.9]$ in Table 2, then the ELI would rise to

$$ELI = 1 - \frac{2 + 2}{3.9 + 3.9} \approx 0.4872$$

When working with fractional payoffs, one immediately encounters the question of grid size – that is, how finely the domain of the payoff space should be partitioned, with strict ordinality at one extreme, and full continuity at the other. In the following section, we assume that continuous payoffs provide a reasonable modeling framework. The question of grid size is discussed in conjunction with the related issue of ties in Section 4.

3.2 Positive Real-Valued Payoffs

To permit games with positive real-valued random payoffs, we employ two randomizations. First, one of the strictly ordinal games, $G(k) \in \mathcal{G}$, is selected with probability 1/144. Second, an arbitrary CDF, $F_X(x)$, $x > 0$ is used to generate twin i.i.d. samples, $\mathbf{X}_1 = [X_{1,1}, X_{1,2}, X_{1,3}, X_{1,4}]$ and $\mathbf{X}_2 = [X_{2,1}, X_{2,2}, X_{2,3}, X_{2,4}]$. This procedure enables us to construct the game $G(k, \mathbf{X})$ with payoffs $\left[a_{11}^{(k,\mathbf{X})}, a_{12}^{(k,\mathbf{X})}, a_{21}^{(k,\mathbf{X})}, a_{22}^{(k,\mathbf{X})}\right]$ and $\left[b_{11}^{(k,\mathbf{X})}, b_{12}^{(k,\mathbf{X})}, b_{21}^{(k,\mathbf{X})}, b_{22}^{(k,\mathbf{X})}\right]$, where

$$a_{ij}^{(k,\mathbf{X})} = X_{1,\left(a_{ij}^{(k)}\right)}$$

$$b_{ij}^{(k,\mathbf{X})} = X_{2,\left(b_{ij}^{(k)}\right)}$$

and $X_{\ell,(m)}$ denotes the m^{th} order statistic of the sample $\mathbf{X}_\ell$.

Essentially, the above approach partitions the set of all possible random games with positive real-valued payoffs into 144 equally weighted subdivisions according to their associated strictly ordinal canonical forms. Letting $ELI(k, \mathbf{X})$ denote the random ELI associated with game $G(k, \mathbf{X})$, we define, for any nonempty subset $\mathcal{H} \subseteq \mathcal{G}$, the conditional random variable

$$ELI_{\mathcal{H}}(k, \mathbf{X}) = ELI(k, \mathbf{X}) \cdot I_{\mathcal{H}}(k)$$

where $I_{\mathfrak{H}}(k)$ is as given previously. This not only allows the structure of Table 6 to be retained – albeit with qualitatively modified definitions of "symmetric" and "constant sum"[6] – but also enhances the efficiency of numerical

[6]A strictly ordinal game $G(k)$ is symmetric if and only if the row player's and column player's payoffs are (1) identical in both cells along one of the game's diagonals, and (2) different in both cells along the other diagonal, whereas a real-valued game $G(k, \mathbf{X})$ is "symmetric" if and only if the *order statistics* of the row player's and column player's payoffs possess properties (1) and (2). Similarly, a strictly ordinal game is constant sum (i.e., a game of pure opposition) if and only if the sum of the row player's and column player's payoffs equals 5 in each of the game's cells, whereas a real-valued game is "constant sum" if and only if the sum of the *order statistics* of the row player's and column player's payoffs equals 5 in each cell.

computations by stratifying the simulation of real-valued games across all canonical subdivisions.

Tables 7 and 8 provide expected values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ for two natural analogues of the strictly ordinal case: real-valued games with $X \sim \text{Uniform}(1,4)$ and $X \sim \text{Uniform}(0,5)$, respectively. By comparing the relative sizes of the $\mathbb{E}[ELI_{\mathcal{H}}(k, \mathbf{X})]$ within each table, one can see that they follow the general pattern of Table 6, with the subdivisions of (a) 0 pure-strategy NCE, and (b) 2 pure-strategy NCE with 0 Pareto-dominant equilibria, manifesting greater inefficiency of NCE. More importantly, however, the tables reveal that the overall inefficiency of NCE, as measured by $\mathbb{E}[ELI(k, \mathbf{X})]$, is very close to that of the strictly ordinal case for $X \sim \text{Uniform}(1,4)$, but noticeably higher for $X \sim \text{Uniform}(0,5)$, despite the fact that both random variables possess exactly the same mean (5/2).

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.1583 (18)	0.0993 (2)	0.1656 (16)	0.1583 (18)
$\|\mathcal{P}\| = 1$	0.0571 (6)	0.0454 (102)	0.1010 (4)	0.0439 (104)	0.0460 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.2095 (3)	0.2132 (6)	N.A. (0)	0.2120 (9)	0.2120 (9)
Total	0.0809 (12)	0.0663 (132)	0.1004 (6)	0.0661 (138)	0.0676 (144)

Table 7: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(1,4)$

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\lvert\mathcal{P}\rvert = 0$	N.A. (0)	0.2422 (18)	0.1624 (2)	0.2522 (16)	0.2422 (18)
$\lvert\mathcal{P}\rvert = 1$	0.0895 (6)	0.0685 (102)	0.1611 (4)	0.0662 (104)	0.0697 (108)
$\lvert\mathcal{P}\rvert = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\lvert\mathcal{P}\rvert = 2$, 0 Pareto Domin.	0.3078 (3)	0.3132 (6)	N.A. (0)	0.3114 (9)	0.3114 (9)
Total	0.1217 (12)	0.1002 (132)	0.1615 (6)	0.0994 (138)	0.1020 (144)

Table 8: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(0, 5)$

3.2.1 Mean-Variance Effects

Such behavior suggests that a mean-preserving, variance-increasing transformation of the random variable X causes $\mathbb{E}[ELI(k, \mathbf{X})]$ to increase. Moreover, Table 9, in which $X \sim \text{Uniform}(0, 3)$, provides evidence for the other half of a mean-variance principle: that is, a variance-preserving, mean-increasing transformation of X causes $\mathbb{E}[ELI(k, \mathbf{X})]$ to decrease.

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.2417 (18)	0.1610 (2)	0.2518 (16)	0.2417 (18)
$\|\mathcal{P}\| = 1$	0.0887 (6)	0.0685 (102)	0.1617 (4)	0.0661 (104)	0.0696 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.3109 (3)	0.3153 (6)	N.A. (0)	0.3138 (9)	0.3138 (9)
Total	0.1220 (12)	0.1002 (132)	0.1615 (6)	0.0995 (138)	0.1021 (144)

Table 9: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(0, 3)$

The above observations may be inferred from the following propositions, which are immediate consequences of (1) and (2).

Proposition 1: If $X = X_0 + \alpha$ for some $X_0 \sim F_{X_0}(x)$, $x > 0$ and constant $\alpha > 0$, then

$$\frac{d\mathbb{E}[ELI(k, \mathbf{X})]}{d\alpha} < 0$$

and $\lim_{\alpha \to \infty} \mathbb{E}[ELI(k, \mathbf{X})] = 0$.

Proposition 2: If $X = \beta X_0$ for some $X_0 \sim F_{X_0}(x)$, $x > 0$ and constant $\beta > 0$, then

$$\frac{d\mathbb{E}[ELI(k, \mathbf{X})]}{d\beta} = 0$$

Note that Proposition 2 implies all differences between the corresponding cells of Tables 8 and 9 are due to simulation errors.

3.2.2 Further Distributional Effects

From Proposition 1, we see that $\mathbb{E}[ELI(k, \mathbf{X})]$ can be made arbitrarily close to its lower bound, 0, simply by adding an arbitrarily large constant to each of the random game's payoffs. This suggests that any meaningful analysis of additional effects of X's distributional shape must rule out unconstrained shifts in the random variable's mean – and more generally, the possibility of an unbounded sample space. Consequently, for exploratory purposes, we will restrict attention to the family of two-parameter Beta (α, β) distributions, which permits a wide variety of probability density function (PDF) morphologies. The insensitivity of $\mathbb{E}[ELI(k, \mathbf{X})]$ to multiplicative scaling (Proposition 2), allows us to set the random variable's sample space as the unit interval, without loss of generality with respect to positive neighborhoods of the origin.[7]

The nine tables below present expected values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ for real-valued games with $X \sim \text{Beta}(\alpha, \beta)$ for systematically varying values of α and β. Tables 10(a,b,c) cover the cases of $\alpha = 0.01$ and $\beta = 0.01$, 1, and 100, respectively, whereas Tables 11(a,b,c) and 12(a,b,c) change α to 1 and 100, respectively, with the same three values of β. Immediately below the title of each table is a brief description of the PDF's shape.

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\lvert\mathcal{P}\rvert = 0$	N.A. (0)	0.3756 (18)	0.2856 (2)	0.3869 (16)	0.3756 (18)
$\lvert\mathcal{P}\rvert = 1$	0.1480 (6)	0.1171 (102)	0.2907 (4)	0.1122 (104)	0.1188 (108)
$\lvert\mathcal{P}\rvert = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\lvert\mathcal{P}\rvert = 2$, 0 Pareto Domin.	0.4736 (3)	0.4636 (6)	N.A. (0)	0.4669 (9)	0.4669 (9)
Total	0.1924 (12)	0.1628 (132)	0.2890 (6)	0.1599 (138)	0.1652 (144)

Table 10(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 0.01)$
(Symmetric PDF with Little Weight in Center)

[7]The Beta (α, β) distribution on $(0, 1)$ is commonly characterized by its PDF, $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $x \in (0, 1)$, for $\alpha > 0$ and $\beta > 0$.

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.9860 (18)	0.9783 (2)	0.9869 (16)	0.9860 (18)
$\|\mathcal{P}\| = 1$	0.4897 (6)	0.4798 (102)	0.9847 (4)	0.4609 (104)	0.4803 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.9875 (3)	0.9848 (6)	N.A. (0)	0.9857 (9)	0.9857 (9)
Total	0.4917 (12)	0.5499 (132)	0.9825 (6)	0.5261 (138)	0.5451 (144)

Table 10(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 1)$
(Moderately Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.9860 (18)	0.9805 (2)	0.9867 (16)	0.9860 (18)
$\|\mathcal{P}\| = 1$	0.4905 (6)	0.4808 (102)	0.9846 (4)	0.4620 (104)	0.4813 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.9864 (3)	0.9857 (6)	N.A. (0)	0.9860 (9)	0.9860 (9)
Total	0.4919 (12)	0.5508 (132)	0.9832 (6)	0.5269 (138)	0.5459 (144)

Table 10(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 100)$
(Highly Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.0005 (18)	0.0007 (2)	0.0005 (16)	0.0005 (18)
$\|\mathcal{P}\| = 1$	0.0002 (6)	0.0002 (102)	0.0006 (4)	0.0002 (104)	0.0002 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.0005 (3)	0.0005 (6)	N.A. (0)	0.0005 (9)	0.0005 (9)
Total	0.0002 (12)	0.0002 (132)	0.0006 (6)	0.0002 (138)	0.0002 (144)

Table 11(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 0.01)$
(Moderately Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.2415 (18)	0.1585 (2)	0.2518 (16)	0.2415 (18)
$\|\mathcal{P}\| = 1$	0.0886 (6)	0.0685 (102)	0.1616 (4)	0.0661 (104)	0.0696 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.3078 (3)	0.3133 (6)	N.A. (0)	0.3115 (9)	0.3115 (9)
Total	0.1213 (12)	0.1001 (132)	0.1606 (6)	0.0993 (138)	0.1019 (144)

Table 11(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 1)$
(Symmetric PDF with Medium Weight in Center; i.e., Uniform PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.4485 (18)	0.3689 (2)	0.4584 (16)	0.4485 (18)
$\|\mathcal{P}\| = 1$	0.1901 (6)	0.1496 (102)	0.3966 (4)	0.1425 (104)	0.1519 (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.5039 (3)	0.5052 (6)	N.A. (0)	0.5048 (9)	0.5048 (9)
Total	0.2210 (12)	0.1997 (132)	0.3874 (6)	0.1934 (138)	0.2015 (144)

Table 11(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 100)$
(Moderately Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\|\mathcal{P}\| = 0$	N.A. (0)	0.0000+ (18)	0.0000+ (2)	0.0000+ (16)	0.0000+ (18)
$\|\mathcal{P}\| = 1$	0.0000+ (6)	0.0000+ (102)	0.0000+ (4)	0.0000+ (104)	0.0000+ (108)
$\|\mathcal{P}\| = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\|\mathcal{P}\| = 2$, 0 Pareto Domin.	0.0000+ (3)	0.0000+ (6)	N.A. (0)	0.0000+ (9)	0.0000+ (9)
Total	0.0000+ (12)	0.0000+ (132)	0.0000+ (6)	0.0000+ (138)	0.0000+ (144)

Table 12(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 0.01)$
(Highly Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\lvert\mathcal{P}\rvert = 0$	N.A. (0)	0.0033 (18)	0.0022 (2)	0.0034 (16)	0.0033 (18)
$\lvert\mathcal{P}\rvert = 1$	0.0011 (6)	0.0010 (102)	0.0018 (4)	0.0010 (104)	0.0010 (108)
$\lvert\mathcal{P}\rvert = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\lvert\mathcal{P}\rvert = 2$, 0 Pareto Domin.	0.0045 (3)	0.0046 (6)	N.A. (0)	0.0046 (9)	0.0046 (9)
Total	0.0017 (12)	0.0014 (132)	0.0019 (6)	0.0014 (138)	0.0014 (144)

Table 12(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 1)$
(Moderately Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
$\lvert\mathcal{P}\rvert = 0$	N.A. (0)	0.0369 (18)	0.0231 (2)	0.0386 (16)	0.0369 (18)
$\lvert\mathcal{P}\rvert = 1$	0.0131 (6)	0.0110 (102)	0.0230 (4)	0.0107 (104)	0.0111 (108)
$\lvert\mathcal{P}\rvert = 2$, 1 Pareto Domin.	0.0000 (3)	0.0000 (6)	N.A. (0)	0.0000 (9)	0.0000 (9)
$\lvert\mathcal{P}\rvert = 2$, 0 Pareto Domin.	0.0499 (3)	0.0510 (6)	N.A. (0)	0.0507 (9)	0.0507 (9)
Total	0.0190 (12)	0.0159 (132)	0.0230 (6)	0.0158 (138)	0.0161 (144)

Table 12(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 100)$
(Symmetric PDF with Much Weight in Center)

These tables show that as the Beta (α, β) distribution becomes increasingly skewed to the right (i.e., the ratio β/α grows larger), the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k, \mathbf{X}\right)\right]$ increase for all NCE subdivisions except the case of 2 pure-strategy NCE with 1 Pareto-dominant equilibrium. This makes intuitive sense because inefficiency will be greater whenever there exists the possibility of an unusually large payoff – that is, a payoff from the right tail that is much greater than more typical payoffs from the "center" of the distribution – entering the 2×2 matrix, thereby raising MP substantially above EP. In fact, as $\beta/\alpha \to \infty$, we find that the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k, \mathbf{X}\right)\right]$ approach limits that are close to: (a) 1.0 for 0 pure-strategy NCE; (b) 0.5 for 1 pure-strategy NCE; (c) 0 for 2 pure-strategy NCE with 1 Pareto-dominant equilibrium; and (d) 1.0 for 2 pure-strategy NCE with 0 Pareto-dominant equilibria. Consequently, $\lim_{\beta/\alpha \to \infty} \mathbb{E}\left[ELI\left(k, \mathbf{X}\right)\right] \approx 9/16 = 0.5625$.[8]

4 Issues of Index Construction

4.1 An Axiomatic Approach?

Our analysis is based upon the NCE or consistent-expectations model of individual behavior. Specifically, we make the simplifying assumption that players select the unique or Pareto-dominant pure-strategy noncooperative equilibrium (NCE) if it exists, and default to the mixed-strategy equilibrium otherwise. Although believing this approach to be reasonable, we recognize it is entirely ad hoc. Therefore, one might well consider whether there exist criteria of an axiomatic nature to specify outcomes of one-shot 2×2 matrix games more rigorously.

Some normative criteria desirable for such a solution concept are listed below. These characteristics may be used (individually or severally) either to define solutions explicitly or to describe and/or assess proposed solutions.

1. Existence. The solution is well defined over the entire domain of 2×2 games. Pure-strategy NCE, which is defined for only 126 of the 144 strictly ordinal games in $\mathcal{G}$, serves as an example of a solution concept that fails to satisfy this property.

[8] From further exploratory simulations based upon the unbounded and heavy-tailed Pareto II (θ, τ) distribution (i.e., $X \sim f_X(x) = \tau\theta^{\tau} / (x + \theta)^{\tau+1}$, $x > 0$) as $\tau \to 0$, we conjecture that $9/16 = 0.5625$ is the "worst-case" upper bound.

2. Uniqueness. The solution is unique for any given game. This is clearly an extremely strong condition, but one that is highly desirable for planning and prediction purposes. Only 108 of the games in $\mathcal{G}$ possess a unique pure-strategy NCE.

3. Symmetry. Two types of symmetry are relevant. Structural symmetry requires that both players have equivalent strategy sets, whereas social symmetry implies that an interchange of player names will not influence the outcome from any solution (e.g., a judge in the Prisoner's Dilemma will apply penalties based upon the players' chosen strategies, irrespective of names). In the set $\mathcal{G}$, only 12 games possess both types of symmetry.

4. Individual rationality. In any game, the solution satisfies the maximin property for both players. Since the games in $\mathcal{G}$ exclude the possibility of ties, all 144 must have a unique individually rational solution.

5. Row or column domination (equivalent to independence from irrelevant alternatives for matrix games). If one row or column of payoffs is dominated by another on a cell-by-cell basis, then there is no individualistic motivation to employ it. Of the games in $\mathcal{G}$, 72 are characterized by either row or column domination alone, whereas 36 are characterized by both.

Although it would seem desirable to employ a solution concept with all five of the above properties, this generally is not possible. (In particular, uniqueness does not hold in matrix games with ties.) However, it may be possible to satisfy all five properties for certain subsets of matrix games, such as $\mathcal{G}$ (which excludes ties). Furthermore, useful solution concepts can be constructed to satisfy various subsets of the five stated properties.

In addition to the above criteria, two well-defined conditions separate schools of thought leaning toward "cooperative solutions" and "noncooperative solutions," respectively:

1. Group rationality (i.e., Pareto optimality).

2. Consistency of prior expectations (i.e., "rational expectations"), or the NCE property.

As a normative criterion, Pareto optimality is highly attractive. This is because "nothing goes to waste" in the sense that no individual in society can improve his or her payoff without decreasing a payoff to someone else. NCE solutions need not satisfy this property, as the NCE solution to the Prisoner's Dilemma readily demonstrates.

4.2 Grid Fineness, Perception, and Ties

As noted previously, the question of grid size (i.e., how finely the domain of the payoff space should be partitioned) arises in the context of fractional payoffs. This issue is driven by two important considerations: (1) the actual minimal proximity of potential payoffs in a given economy; and (2) the ability of players to distinguish, cognitively and/or emotionally, between payoff levels that are numerically quite close.

In Section 3.2, we worked with a continuum of payoffs, effectively assuming payoff amounts can be arbitrarily close and/or practically indistinguishable. However, the reasonableness of this assumption requires further study. In particular, it is known that ties appear in many economic settings, warranting a careful assessment of their omission.

4.3 Games with More than 2 Strategies?

A fundamentally different approach to the study of two-person games would permit each player to possess an arbitrary number of strategies, $s \geq 2$. In the simplest case, one could define a strictly ordinal $s \times s$ game by assigning payoffs with integer values $1, 2, \ldots, s^2$ to each player. For this game, the value of the maximum joint payoff, MP, would take on an integer value from $s+1$ to $2s$.

If one were to select the payoffs randomly, without replacement, from $\{1, 2, \ldots, s^2\}$, then the distribution of MP would shift to the right for increasing s, consistent with the observations of footnote 5. This rightward shift is apparent even for $s = 2$ in the case of continuous payoffs, X, drawn from the Uniform $(0, 1)$ distribution. Figure 1 shows the relevant PDF of MP.

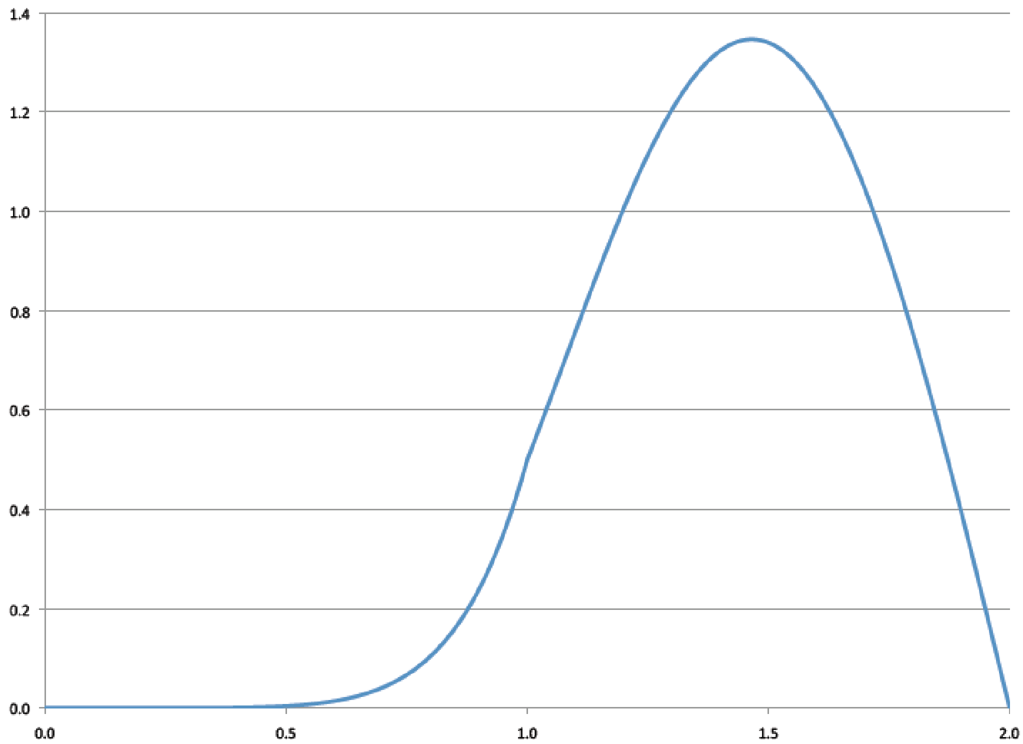

Figure 1. Probability Density Function of MP for
$X \sim \text{Beta}(\alpha = 1, \beta = 1) \equiv \text{Uniform}\,(0, 1)$

5 Games with Low Information about Opponent Payoffs

In Sections 3 and 4 we studied the ELI in the context of games with high information, in which each player knows the precise payoff matrix of his or her opponent. However, in a large society it is more reasonable to expect many interactions to occur with unknown individuals, in which case the inefficiency of NCE is likely to be greater. In the present section, we therefore consider random 2×2 games in which each player knows his/her own payoffs, but not those of the opponent. This type of game may be used to model an individual interacting anonymously with the rest of society.

To study such low-information games, we will use the same two-stage randomization process described above for generating real-valued games, but model player behavior quite differently. Since each player knows that he/she may encounter a real-valued game associated with any of the canonical games in $\mathcal{G}$, but is told only his/her own payoffs, then the NCE is simply to select the strategy that maximizes the player's expected payoff under the assumption that the opponent will play a 50/50 mixed strategy. This assumption is justified by the fact that, in NCE, the row player's best response yields a 50/50 probability from the perspective of the column player, and vice versa. Another way to think of this is that each player knows that, under the randomized-game procedure, he/she is essentially playing against a stochastic doppelganger, from which overall symmetry can be deduced.

Tables 13-15 summarize the expected values of the $ELI_{\mathcal{H}}(k, \mathbf{X})$ for low-information random games with $X \sim \text{Uniform}(1,4)$, $X \sim \text{Uniform}(0,5)$, and $X \sim \text{Uniform}(0,3)$, respectively. Although these tables correspond exactly with Tables 7-9 of Section 3.2, the row-subdivision titles are now placed in quotation marks because the original NCE subdivisions of $\mathcal{G}$ do not manifest themselves in the actual play of the low-information games.

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.1240 (18)	0.0849 (2)	0.1289 (16)	0.1240 (18)
" $\|\mathcal{P}\| = 1$"	0.1298 (6)	0.1381 (102)	0.0925 (4)	0.1393 (104)	0.1376 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.2352 (3)	0.2123 (6)	N.A. (0)	0.2199 (9)	0.2199 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.1789 (3)	0.2043 (6)	N.A. (0)	0.1958 (9)	0.1958 (9)
Total	0.1684 (12)	0.1425 (132)	0.0900 (6)	0.1471 (138)	0.1447 (144)

Table 13: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(1,4)$, Low-Information Games

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.1878 (18)	0.1352 (2)	0.1944 (16)	0.1878 (18)
" $\|\mathcal{P}\| = 1$"	0.1913 (6)	0.2038 (102)	0.1451 (4)	0.2054 (104)	0.2031 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.3331 (3)	0.3035 (6)	N.A. (0)	0.3134 (9)	0.3134 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.2643 (3)	0.3012 (6)	N.A. (0)	0.2889 (9)	0.2889 (9)
Total	0.2450 (12)	0.2106 (132)	0.1418 (6)	0.2166 (138)	0.2135 (144)

Table 14: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(0, 5)$, Low-Information Games

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.1881 (18)	0.1370 (2)	0.1945 (16)	0.1881 (18)
" $\|\mathcal{P}\| = 1$"	0.1898 (6)	0.2039 (102)	0.1455 (4)	0.2053 (104)	0.2031 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.3330 (3)	0.2995 (6)	N.A. (0)	0.3107 (9)	0.3107 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.2629 (3)	0.3000 (6)	N.A. (0)	0.2877 (9)	0.2877 (9)
Total	0.2439 (12)	0.2105 (132)	0.1427 (6)	0.2163 (138)	0.2132 (144)

Table 15: Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Uniform}(0, 3)$, Low-Information Games

Comparing Tables 13-15 with their earlier counterparts, we see that $\mathbb{E}\left[ELI\left(k,\mathbf{X}\right)\right]$ is substantially greater in each case, a feature primarily attributable to increases in the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k,\mathbf{X}\right)\right]$ for the subdivisions of (a) 1 pure-strategy NCE, and (b) 2 pure-strategy NCE with 1 Pareto-dominant equilibrium. Overall, these increases cause the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k,\mathbf{X}\right)\right]$ to vary much less across the individual subsets $\mathcal{H}\subset\mathcal{G}$, which is not surprising because the row and column subdivisions are less relevant when each player views his/her opponent as playing a 50/50 mixed strategy. We note further that Tables 13-15 are entirely consistent with the mean-variance analysis of Section 3.2.1, and that Propositions 1 and 2 continue to hold for the low-information games.

In the Appendix, we include nine additional tables (A.1(a,b,c)-A.3(a,b,c)) providing low-information counterparts to Tables 10(a,b,c)-12(a,b,c) of Section 3.2 (based upon the Beta (α,β) distribution). In addition to possessing a substantially greater $\mathbb{E}\left[ELI\left(k,\mathbf{X}\right)\right]$ – caused by increases in the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k,\mathbf{X}\right)\right]$ for the same NCE subdivisions as in the uniform-distribution case – we find that, as $\beta/\alpha\to\infty$, the $\mathbb{E}\left[ELI_{\mathcal{H}}\left(k,\mathbf{X}\right)\right]$ approach limits that are close to: (a) 0.625 for 0 pure-strategy NCE; (b) 0.625 for 1 pure-strategy NCE; (c) 0.5 for 2 pure-strategy NCE with 1 Pareto-dominant equilibrium; and (d) 0.75 for 2 pure-strategy NCE with 0 Pareto-dominant equilibria. Consequently, $\lim_{\beta/\alpha\to\infty}\mathbb{E}\left[ELI\left(k,\mathbf{X}\right)\right]\approx 5/8=0.625$.

6 Discussion

6.1 All Possible Worlds or Our World?

In the present work, we have focused on selections from the set of 144 games in $\mathcal{G}$. These represent the closed set of all possible worlds for 2×2 matrix games with strictly ordered preferences, and thus enabled us to explore, statistically, all strategic structures of this size and type.

We must recognize, however, that an exercise useful for one purpose may be highly misleading for another. In particular, averages do not tell much about specifics. Therefore, in conjunction with our study of the average inefficiency of all games in $\mathcal{G}$, we must note that certain particular game structures are more conducive to efficiency than others under the same behavioral conditions.

Scientific evidence suggests that most organisms are constructed to live in highly restricted environments, at least in the short run, and even human beings are not designed to adjust immediately to all strategic structures. Under any metric that one can contrive, life as we know it would appear to be viable only within a highly restricted bandwidth of environments. Given our emphasis on features of NCE averaged over all strategic environments, it is important, for both theory and applications, to ask which subsets of $\mathcal{G}$ are most representative of problems set in economic, political, social, or other contexts.

Of the 144 games in $\mathcal{G}$, the Prisoner's Dilemma has attracted the greatest research attention by far, with the Stag Hunt and Chicken coming second and third, respectively. Is this because these games are more central to economic behavior than other games? or is it simply that they pose interesting puzzles, and are not necessarily representative of economic life? These are open questions whose answers appear to depend parametrically on the structure of preferences. Taking an agnostic approach to such questions – which may prove stubbornly unanswerable in many contexts – we appeal to a simple indifference principle to justify the use of random games.

6.2 Value of Coordination, Control, and Governance

Estimates of the cost of governance over history and place range anywhere from 5 to 60 percent of GNP (see Shubik, 2011). At the level of the human organism, the energy use of the brain has been estimated at about 20 percent of an individual's total consumption (see Mink, Blumenschine, and Adams, 1981).

In the context of strictly ordinal 2×2 matrix games with high payoff information, Table 16 shows that the overall ELI varies considerably, from 0 to 56.25 percent, where the latter value is achieved for highly right-skewed payoff distributions. For games with low payoff information, Table 17 shows that the overall index becomes much greater, on a percentage basis, for left-skewed and symmetric payoff distributions, but attains a maximum of only 60.5 percent for highly right-skewed payoffs. Applying the indifference principle to the payoff distribution within each particular game, it is interesting to note that the overall ELI is approximately 10.19 percent for high-information games with payoffs that are uniform on $(0, 1)$, and approximately 21.32 percent for low-information games of the same sort.

	$\beta = 0.01$	$\beta = 1.00$	$\beta = 100$	$\beta = \infty$
$\alpha = 0.01$	0.1652	0.5451	0.5459	0.5625
$\alpha = 1.00$	0.0002	0.1019	0.2015	0.5625
$\alpha = 100$	0.0000+	0.0014	0.0161	0.5625
$\alpha = \infty$	0.0000	0.0000	0.0000	

Table 16. Expected Values of Overall $ELI\,(k, \mathbf{X})$ for Various Choices of α and β, High-Information Games

	$\beta = 0.01$	$\beta = 1.00$	$\beta = 100$	$\beta = \infty$
$\alpha = 0.01$	0.2865	0.6086	0.6094	0.6250
$\alpha = 1.00$	0.0051	0.2132	0.3110	0.6250
$\alpha = 100$	0.0001	0.0041	0.0351	0.6250
$\alpha = \infty$	0.0000	0.0000	0.0000	

Table 17. Expected Values of Overall $ELI\,(k, \mathbf{X})$ for Various Choices of α and β, Low-Information Games

Overall, we observe that activities of coordination, information dissemination, feedback analysis, and the development of appropriate behavioral constraints can come at a high cost that is likely to depend parametrically on various characteristics of the problem considered. Moreover, we note that our analysis attempts only to estimate the possible value of coordination and control, omitting any discussion of the related problem of designing efficient mechanisms to render efficient governance.

7 Conclusions

In most areas of economics, money is a reasonably useful – though far from perfect – measure of resource allocation. In the context of 2×2 matrix games, we treated the sum of the row and column players' payoffs as a monetary measure of social welfare, and proposed a novel efficiency loss index (ELI), to calculate the inefficiency of NCE in contrast with fully cooperative solution concepts. Essentially, this index allows one to analyze the amount players would be willing to pay a governing referee to coordinate the outcome of a game.

Under a principle of indifference among "all possible worlds," we first calculated the average *ELI* for a random draw from the set of all strictly ordinal 2×2 games, and obtained a relatively low value of 6.53 percent. However, by cardinalizing the payoffs of the strictly ordinal game, we were able to see that this measure is sensitive to both the distribution of payoffs and payoff-information conditions. In particular, the *ELI* increases to 10.19 percent for high-information payoffs that are uniform on $(0, 1)$, and 21.32 percent for low-information payoffs with the same distribution. For highly left-skewed payoffs, the index approaches 0 in both high- and low-information games, whereas for highly right-skewed payoffs, it approaches 56.25 and 62.5 percent in high- and low-information games, respectively.

At least in terms of metaphor, these results suggest that if we consider all worlds without selection bias, the lower bound on the value of government is somewhat greater than 5 percent, and the upper bound somewhat less than 65 percent. Over most of this spectrum, low information accounts for much of the need for coordination.

8 References

Barany, I., Lee, J., and Shubik, M., 1992, Classification of Two-Person Ordinal Bimatrix Games, *International Journal of Game Theory*, 27, 267-290.

Borm, P., 1987, A Classification of 2×2 Bimatrix Games, *Cahiers du C.E.R.O.*, 29, 1-2, 69-84.

Kilgore, D. M. and Frazer, N. M., 1988, A Taxonomy of All Ordinal 2×2 Games, *Theory and Decision*, 24, 99-117.

Halpern, J. Y., 2003, A Computer Scientist Looks at Game Theory, *Games and Economic Behavior*, 45, 114-131.

Harsanyi, J. C. and Selten, R., 1988, A General Theory of Equilibrium Selection in Games, MIT Press, Cambridge, MA.

Mink, J. W., Blumenschine, R. J., and Adams, D. B., 1981, Ratio of Central Nervous System to Body Metabolism in Vertebrates: Its Constancy and Functional Basis, *American Journal of Physiology*, 241, 3, R203-212.

Powers, M. R., 2012, *Acts of God and Man: Ruminations on Risk and Insurance*, Columbia University Press, New York.

Powers, M. R. and Shen, Z., 2008, Social Stability and Catastrophe Risk: Lessons from the Stag Hunt, *Journal of Theoretical Politics*, 20, 4, 477-497.

Rapoport A. and Guyer, M., 1966, *The 2X2 Game*, University of Michigan Press, Ann Arbor, MI.

Rapoport A., Guyer, M., and Gordon, R. G., 1966, A Taxonomy of the 2×2 Game, *General Systems*, 11, 203-214.

Robinson D. and Goforth, D., 2005, *The Topology of 2X2 Games: A New Periodic Table*, Routledge, London.

Rothblum, U., 2006, Bounding the Inefficiency of Nash Equilibria in Games with Finitely Many Players, *Operations Research Letters*, 15, 700-706.

Roughgarden, T., 2009, Intrinsic Robustness of the Price of Anarchy, in Proceedings of the 41st ACM Symposium on Theory of Computing (STOC), 513-522.

Shubik, M., 2011, *The Theory of Money ad Financial Institutions, Volume 3*, MIT Press, Cambridge, MA.

Appendix

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.2530 (18)	0.2012 (2)	0.2595 (16)	0.2530 (18)
" $\|\mathcal{P}\| = 1$"	0.2684 (6)	0.2764 (102)	0.2226 (4)	0.2780 (104)	0.2760 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.3931 (3)	0.3614 (6)	N.A. (0)	0.3719 (9)	0.3719 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.3736 (3)	0.4057 (6)	N.A. (0)	0.3950 (9)	0.3950 (9)
Total	0.3259 (12)	0.2830 (132)	0.2155 (6)	0.2896 (138)	0.2865 (144)

Table A.1(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 0.01)$, Low-Information Games (Symmetric PDF with Little Weight in Center)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.6072 (18)	0.6021 (2)	0.6079 (16)	0.6072 (18)
" $\|\mathcal{P}\| = 1$"	0.6129 (6)	0.6072 (102)	0.6660 (4)	0.6053 (104)	0.6075 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.4950 (3)	0.4964 (6)	N.A. (0)	0.4959 (9)	0.4959 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.7400 (3)	0.7351 (6)	N.A. (0)	0.7367 (9)	0.7367 (9)
Total	0.6152 (12)	0.6080 (132)	0.6447 (6)	0.6070 (138)	0.6086 (144)

Table A.1(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 1)$, Low-Information Games (Moderately Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.6101 (18)	0.6123 (2)	0.6098 (16)	0.6101 (18)
" $\|\mathcal{P}\| = 1$"	0.6117 (6)	0.6078 (102)	0.6653 (4)	0.6058 (104)	0.6080 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.5002 (3)	0.4987 (6)	N.A. (0)	0.4992 (9)	0.4992 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.7353 (3)	0.7356 (6)	N.A. (0)	0.7355 (9)	0.7355 (9)
Total	0.6148 (12)	0.6089 (132)	0.6477 (6)	0.6078 (138)	0.6094 (144)

Table A.1(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 0.01, \beta = 100)$, Low-Information Games (Highly Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.0053 (18)	0.0052 (2)	0.0053 (16)	0.0053 (18)
" $\|\mathcal{P}\| = 1$"	0.0049 (6)	0.0051 (102)	0.0069 (4)	0.0050 (104)	0.0051 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.0085 (3)	0.0036 (6)	N.A. (0)	0.0052 (9)	0.0052 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.0017 (3)	0.0063 (6)	N.A. (0)	0.0048 (9)	0.0048 (9)
Total	0.0050 (12)	0.0051 (132)	0.0063 (6)	0.0050 (138)	0.0051 (144)

Table A.2(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 0.01)$, Low-Information Games (Moderately Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.1878 (18)	0.1374 (2)	0.1942 (16)	0.1878 (18)
" $\|\mathcal{P}\| = 1$"	0.1901 (6)	0.2036 (102)	0.1459 (4)	0.2050 (104)	0.2028 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.3313 (3)	0.3032 (6)	N.A. (0)	0.3125 (9)	0.3125 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.2641 (3)	0.3014 (6)	N.A. (0)	0.2890 (9)	0.2890 (9)
Total	0.2439 (12)	0.2104 (132)	0.1431 (6)	0.2163 (138)	0.2132 (144)

Table A.2(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 1)$, Low-Information Games (Symmetric PDF with Medium Weight in Center; i.e., Uniform PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.2828 (18)	0.2538 (2)	0.2864 (16)	0.2828 (18)
" $\|\mathcal{P}\| = 1$"	0.2979 (6)	0.3001 (102)	0.2754 (4)	0.3009 (104)	0.3000 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.4033 (3)	0.3897 (6)	N.A. (0)	0.3942 (9)	0.3942 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.4040 (3)	0.4226 (6)	N.A. (0)	0.4164 (9)	0.4164 (9)
Total	0.3508 (12)	0.3074 (132)	0.2682 (6)	0.3129 (138)	0.3110 (144)

Table A.2(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 1, \beta = 100)$, Low-Information Games (Moderately Right-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.0001 (18)	0.0000+ (2)	0.0001 (16)	0.0001 (18)
" $\|\mathcal{P}\| = 1$"	0.0001 (6)	0.0001 (102)	0.0001 (4)	0.0001 (104)	0.0001 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.0001 (3)	0.0000+ (6)	N.A. (0)	0.0001 (9)	0.0001 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.0000+ (3)	0.0001 (6)	N.A. (0)	0.0000+ (9)	0.0000+ (9)
Total	0.0001 (12)	0.0001 (132)	0.0001 (6)	0.0001 (138)	0.0001 (144)

Table A.3(a): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 0.01)$, Low-Information Games (Highly Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.0037 (18)	0.0028 (2)	0.0038 (16)	0.0037 (18)
" $\|\mathcal{P}\| = 1$"	0.0036 (6)	0.0039 (102)	0.0031 (4)	0.0039 (104)	0.0039 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.0072 (3)	0.0057 (6)	N.A. (0)	0.0062 (9)	0.0062 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.0039 (3)	0.0055 (6)	N.A. (0)	0.0050 (9)	0.0050 (9)
Total	0.0046 (12)	0.0040 (132)	0.0030 (6)	0.0041 (138)	0.0041 (144)

Table A.3(b): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 1)$, Low-Information Games (Moderately Left-Skewed PDF)

	"Symmetric"	"Asymmetric"	"Constant Sum"	"Non-Const. Sum"	Total
" $\|\mathcal{P}\| = 0$"	N.A. (0)	0.0292 (18)	0.0202 (2)	0.0303 (16)	0.0292 (18)
" $\|\mathcal{P}\| = 1$"	0.0319 (6)	0.0335 (102)	0.0218 (4)	0.0339 (104)	0.0334 (108)
" $\|\mathcal{P}\| = 2$, 1 Pareto Domin."	0.0594 (3)	0.0544 (6)	N.A. (0)	0.0561 (9)	0.0561 (9)
" $\|\mathcal{P}\| = 2$, 0 Pareto Domin."	0.0420 (3)	0.0487 (6)	N.A. (0)	0.0465 (9)	0.0465 (9)
Total	0.0413 (12)	0.0346 (132)	0.0213 (6)	0.0357 (138)	0.0351 (144)

Table A.3(c): Expected Values of $ELI_{\mathcal{H}}(k, \mathbf{X})$ Associated with $\mathcal{H} \subseteq \mathcal{G}$ for $X \sim \text{Beta}(\alpha = 100, \beta = 100)$, Low-Information Games (Symmetric PDF with Much Weight in Center)