On Singularities in Combination-Driven Models of Technological Innovation

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It has been suggested that innovations occur mainly by combination: the more inventions accumulate, the higher the probability that new inventions are obtained from previous designs. Additionally, it has been conjectured that the combinatorial nature of innovations naturally leads to a singularity: at some finite time, the number of innovations should diverge. Although these ideas are certainly appealing, no general models have been yet developed to test the conditions under which combinatorial technology should become explosive. Here we present a generalised model of technological evolution that takes into account two major properties: the number of previous technologies needed to create a novel one and how rapidly technology ages. Two different models of combinatorial growth are considered, involving different forms of ageing. When long-range memory is used and thus old inventions are available for novel innovations, singularities can emerge under some conditions with two phases separated by a critical boundary. If the ageing has a characteristic time scale, it is shown that no singularities will be observed. Instead, a "black hole" of old innovations appears and expands in time, making the rate of invention creation slow down into a linear regime.

Keywords: Technology, evolution, combination, networks, patents, phase transitions

1. Introduction

Technology is one of the most obvious outcomes of human culture. Over the last 100.000 years, humans have been able to manipulate their environments and the species they used to interact with in an amazing range of ways. Technological inventions have been developing at an accelerated rate since the industrial revolution [1-5] and economist Brian Arthur conjectured that such rapid growth is a consequence of the underlying dynamics of combination that drives the process [2]. Specifically, it has been suggested that novelties arise mainly as a consequence of new forms of interaction between previous artifacts or inventions [2]. Such view connects the pace of man-made evolutionary designs with a basic principle of biological evolution: the presence of tinkering [6] as a dominant way of generating new structures [7,8].

Systematic studies of technological change are difficult to perform due to a number of problems. These include the lack of a genome-like description of artifacts and the complex nature of their design paths. The study (largely naturalistic) of some particular systems, such as cornets [9-10] reveals some interesting similarities, while uncovering deep differences with cultural change. More recent work based on network theory [11] provided a novel quantitative approach to technological change
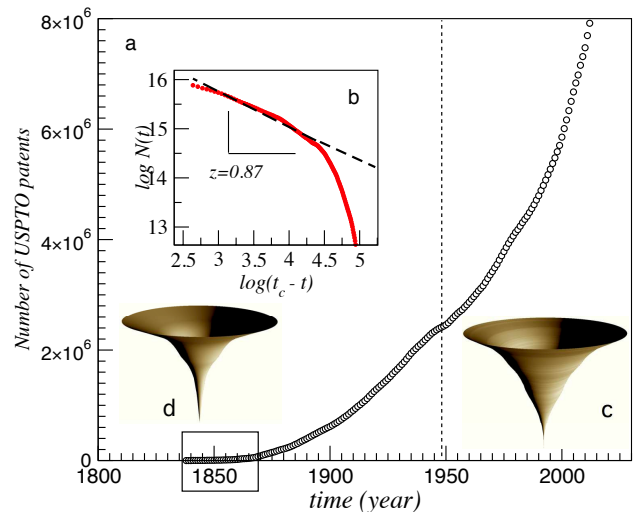
Figure 1. Hyperbolic growth of the total USPTO dataset (a). In (b) we plot of the logarithm of patent counts against $t_c - t$ (b) for $t > 1950$. The best fit for this data set gave an estimate $\eta = 0.87 \pm 0.06$ consistent with an exponent $z \approx 1.14$. The inset (c) provides a picture of the spindle diagram for $N(t)$ and in (d) we show the same diagram for the initial part of the time series.

that defines a formal framework to explore technological change and the impact of design principles.

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One consequence of the combination principle proposed by Arthur is that the growth dynamics of inventions would be faster than exponential (or super-Malthusian) and should exhibit a finite-time singularity [12]. The implications of such rapidly accelerating innovation processes have been discussed in recent years, raising controversial speculations [13]. Predicting the progress of technological change is a timely issue but also a difficult task. Nevertheless, some insights have been gathered from using proper databases and statistical methods [14,15].

A surrogate of the ways in which innovations take place in time is provided by patent files [16-18]. Patents are well-defined objects introducing a novel design, method, or solution for a given problem or set of problems. Existing data bases store multiple levels of patent description and they can be analyzed in full detail. Additionally, they indicate what previous novelties have been required to build new ones. An example is given by the U.S. Patent and Trademark Office (USPTO) patents filed between 1835 and 2010 ‡. In figure 1a we display the total number N of filed patents, which clearly reveals an accelerated trend over time [3,12]. The dashed line in particular indicates the start of the modern information technology era (around 1950).

In [18] a study of the USPTO database was made in search for evidence of combinatorial evolution. The authors concluded that truly new technological capabilities are slowing down in their rate of appearance, but nevertheless a great deal of combination is present thus allowing for a "practically infinite space of technological configurations" [18]. One potential outcome of this virtually exploding space is a growth dynamics displaying potential singularities, i. e. divergent numbers of inventions would eventually occur as we approach a finite time window. In this paper we want to address the problem of how to define the conditions for technological singularities to be expected. Two main components of combinatorial dynamics will be taken into account: (a) the diversity (number) of potential innovations required to trigger a new one and (b) the degree of ageing that makes older innovations less likely to be used. As shown below, two main phases are expected in this diversity-aging space, defining the conditions for singularities to be present.

2. Hyperbolic dynamics: minimal model

Using the simplest approximation, we assume a neutral model of innovation based on pairwise combinations of existing designs. In this model, the set $\pi = \{\pi_1, \pi_2, ..., \pi_{n(t)}\}$ defines the "design universe" at any given time t. Each $\pi_j \in \pi$ represents an invention as described, for example, by a patent file. Here, $N(t)$ is the total number of inventions (patents) at year t. Two given designs will then combine at a given time t with a given probability: $\pi_i + \pi_j \xrightarrow{\mu_{ij}} \pi_{n(t)+1}$ where μ_{ij} weights the likelihood of such an event to happen. This defines a second-order (bimolecular) reaction kinetics [19-21]. Such nonlinear reaction dynamics seems to pervade

‡http://patft.uspto.gov

the super-exponential growth observed in a number of economic and demographic systems [22-24].
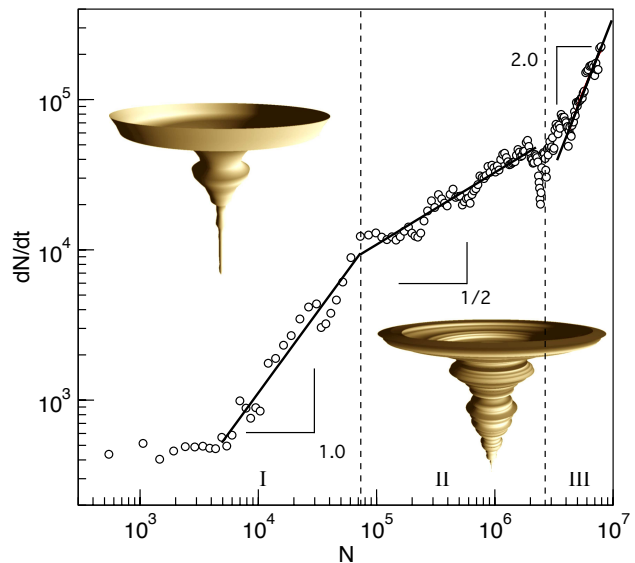

Figure 2. Different rates of patent growth are observed in the record of USA patents. Here we plot the rate of patent numbers growth dN/dt as a function of N. Here we estimate the exponents associated to the scaling $dN/dt \sim N^\beta$ for three different domains that appear to characterise this data set (here indicated as I, II and III). The three exponents $\beta = 1, 1/2, 3/2$ indicate different kinetic phases of innovation. The two spindle diagrams (insets) correspond to the first (I) interval (left) and the whole time series (right)

How is this space expanded? We will assume that every element $\pi_j \in \pi$ has the same potential to attach to other existing elements. We can consider a more complex kinetic equation, namely:

$$\frac{dN}{dt} = \mu(t)N^{1+z} \tag{1}$$

where $\mu(t)$ is the attachment rate of inventions at a given time. The parameter $0 \le z \le 1$ weights the departure from the linear scenario. For $z = 0$ the above yields to exponential growth. If a pure (pairwise) combination scenario were at work, we would see $z = 1$, since any pair of inventions is likely to interact. By solving equation (1) we obtain

$$N(t) = \left[N^{-z}(0) - z \int_0^t \mu(\tau)d\tau \right]^{-\frac{1}{z}} \tag{2}$$

For the simplest scenario where μ can be considered constant, i. e. $\mu = \langle \mu(t) \rangle = (\int_0^t \mu(\tau)d\tau)/t$, we can write the previous equation as follows:

$$N(t) = (z\mu)^{-1/z} \left(t_s - t \right)^{-1/z} \tag{3}$$

where t_s is a finite time given by $t_s = 1/(z\mu N_0^z)$ with $N_0 = N(0)$. A very interesting feature of this solution is the presence of a *singularity*: as we approach t_s, a divergence occurs in N. For the pure combination solution with $z = 1$ we would observe a growth curve

following:

$$N(t) = \frac{1}{\mu}\left(\frac{1}{t_s - t}\right) \qquad (4)$$

which provides a prediction of how invention numbers will increase under a neutral model where all inventions (patents) are equally likely to combine.

In order to fit the last part ($t > 1980$) of the USPTO data set (part III of the time series in figure 2) with the hyperbolic growth prediction and estimate the parameters involved in defining it, we will make a logarithmic transformation of the previous general equation as [25]:

$$\ln N(t) = \ln \Gamma + \eta \ln(t_s - t) \qquad (5)$$

where $\Gamma = (z\mu)^{-1/z}$ and $\eta = -1/z$. Using this transformation, we can estimate the set of parameters associated to the growth process. Following [25], we fixed the critical time t_s to a given year value and fitted the remaining two (free) parameters. By using different t_s values, it is possible to determine the best fitting. For our data set (figure 1, inset), we obtain a critical time $t_s = 2027$ and $\eta = 0.87 \pm 0.09$. This gives $z \approx 1.14$ and it would thus approach the expected $\dot{N} \sim N^2$ hyperbolic law[§].

Is this a general result? An interesting observation comes from plotting the rate of patent generation (dN/dt) against the total number of patents N at a given time. This is displayed in figure 2, where the USPTO data reveal three domains of scaling. These correspond to (I) the industrial revolution (II) the historical period between the end of the IR and the 1990s and (III) the interval ($1990 - 2010$). The last part (III) is associated with the aftermath of the information technology revolution associated to the rise of Internet.

The three domains seem to fit to three different types of kinetics, i e. we would have, following the previous scheme, three different dynamical laws, with $dN/dt = \mu N^\beta$ characterised by exponents $\beta_I \sim 1.0$, $\beta_{II} \sim 0.5$ and $\beta_{III} \sim 2.0$. For the later domain, we used the last part of the time series from 1980 to the end of the time series. The reason for this choice is the presence of a drop in the time series of patent production that took place due to new management policies by USPTO (12). The last part seems consistent with the minimal hyperbolic model, whereas the first two involve smaller exponents.

Of note, we can see that the expected linear law associated to the patent dynamics within the IR implies that technological change would lead to exponential growth, whereas the second domain (which includes two World Wars and the great depression) suggests a sublinear dynamical growth equation. Two questions emerge from this seemingly diverse phases. One is whether or not a singularity should be expected and the second the meaning of the scaling exponent β. As will be shown below, a generalised model taking into account two key features of the innovation process provide answers to the previous questions.

[§]However deciding what part of the time series to take for the interpolation is far from simple. Several economic, technological and even patent-office related factors influence some of the rapid changes found in the USPTO time series.

3. Generalized models

If our assumptions were correct, A critical horizon is thus obtained, which indicates that a singularity appears to exist around year 2027. Such prediction is performed by using the last part of the time series. However, accelerated patterns of innovation growth can be seen at different time intervals, showing different exponents. How can we explain these differences? When are singularities expected to occur under these different regimes?

Several simplifications have been made in the model above. One is that a limited number of previous innovations are combined among them to obtain a new one. Another is that we assumed by default that all innovations can contribute to future technologies, when actually many will become obsolete and get replaced. Some type of ageing factor needs to be considered. Such ageing has been found to be present in different types of growing networks [26-28] including different forms collaboration among scientists and links among innovations [17] and will be also studied here.

One way of including multiple innovations is to consider the average number k of inventions that are used to obtain new ones. On the other hand, we need to define the way two elements might interact. This can be done by considering a generalised integral equation:

$$\frac{dN}{dt} = \int_0^N \cdot \overset{k}{\cdots} \cdot \int_0^N \Gamma(\tau_1, ..., \tau_k)d\tau_1...d\tau_k \qquad (6)$$

Here the kernel $\Gamma(\tau_1, ..., \tau_k)$ defines the likelihood that k different patents interact in order to give a new invention. This general expression contains the hyperbolic scenario introduced above as one special case when all elements can equally interact and thus $\Gamma = \mu$. To see this, notice that we have now

$$\frac{dN}{dt} = \int_0^N \int_0^N \mu d\tau_1 d\tau_2$$

$$= \mu \left(\int_0^N d\tau_1\right)\left(\int_0^N d\tau_2\right) = \mu N^2 \qquad (7)$$

From now on we will assume that the kernel can be factorized: all inventions can interact in similar ways and thus

$$\Gamma(\tau_1, ..., \tau_k) = \prod_{l=1}^{k} \Gamma(\tau_l) \qquad (8)$$

In that case, the previous equation reads now:

$$\frac{dN}{dt} = \int_0^N \cdot \overset{k}{\cdots} \cdot \int_0^N \prod_{l=1}^{k} \Gamma(\tau_l)d\tau_1...d\tau_k \qquad (9)$$

and thus our general model to be explored below reads now:

$$\frac{dN}{dt} = \prod_{l=1}^{k}\left[\int_0^N \Gamma(\tau_l)d\tau_l\right] \qquad (10)$$
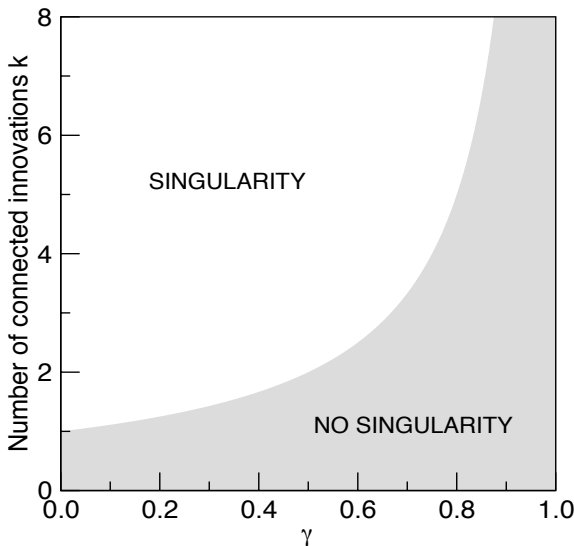
Figure 3. Two phases predicted by the generalised model of technological evolution with power law ageing. The white area correspond to all parameter combinations allowing a singularity to emerge through a hyperbolic growth.

3.1. *Power law aging*

This integral equation contains the number of innovations required to further expand the technological space. Now we need to introduce how ageing affects the range of interactions. One choice is a power law Kernel, namely

$$\Gamma(\tau_l) \sim \mu^{1/k}(N - \tau_l)^{-\gamma} \qquad (11)$$

where the scaling exponent $\gamma \geq 0$ gives a measure of how fast previous innovations become obsolete and are not incorporated. This Kernel has been used in different contexts, including in the analysis of collaborations among researchers, which is a closely related problem (15). In this case, the general model is written as

$$\frac{dN}{dt} = \prod_{l=1}^{k} \left[\int_0^N \mu^{1/k}(N - \tau_l)^{-\gamma} d\tau_l \right] \qquad (12)$$

If we assume equivalence between all the components of our system, all kernels being equal we obtain here:

$$\frac{dN}{dt} = \left[\int_0^{N(t)} \mu^{1/k}(N - \tau)^{-\gamma} d\tau \right]^k$$
$$= \frac{\mu}{(1 - \gamma)^k} N^{(1-\gamma)k} \qquad (13)$$

By solving this equation, we can show that the solution reads:

$$N(t) = [C_0 + \eta(k, n)t]^{1/(1-(1-\gamma)k)} \qquad (14)$$

The constants are defined by $C_0 = N_0^{1-(1-\gamma)k}$ and

$$\eta(k, n) = \frac{1 - (1-\gamma)k}{(1-\gamma)^k} \qquad (15)$$

respectively. This equation will be consistent with a singularity provided that the scaling exponent is negative.

This leads to a critical condition:

$$k > k_c = \frac{1}{1 - \gamma} \qquad (16)$$

The phase diagram predicted by this critical boundary is shown in figure 3, where we plot $k_c(\gamma)$. The two domains showing or lacking a singularity are separate by this curve. As we can see, singularities are expected even for $\gamma = 0$ provided that $k > 1$. Similarly, when $k = 2$ we have the standard pairwise reaction scheme described above

3.2. *Exponential aging*

The power law kernel introduces a long tail and thus long-memory effects. Although links with old inventions are much rare, they can be established and thus a contribution will always be expected. What is the impact of using a different type of interaction Kernel involving a more rapid decay that forbids new inventions to "connect" with very old ones?. This can be modelled with an exponential decay of the form $e^{-\mu(N-\alpha)}$. As defined, the smaller the value of μ, the longer the age (namely, $N - \alpha$) that patents can reach while being still able to generate new inventions. Indeed, in the limit $\mu \to 0$ all the inventions would equally contribute (no matter their age), recovering again the simple hyperbolic (pairwise) scenario analyzed in section 2.

In order to illustrate the impact of this limited memory, let us consider again the pairwise $(k = 2)$ scenario. The generation of new patents is now given by

$$\frac{dN}{dt} = \mu \prod_{l=1}^{2} \left[\int_0^N e^{-\gamma(N-\tau_l)} d\tau_l \right]$$
$$= \mu' \left[e^{-\gamma N} - 1 \right]^2 \qquad (17)$$

where $\mu' = \mu/\gamma^2$. By solving this equation we obtain an implicit form:

$$(N - N_0) + \frac{1}{\gamma g_\gamma(N)} - \frac{1}{\gamma g_\gamma(N_0)} +$$
$$+ \frac{1}{\gamma} \ln \left[\frac{g_\gamma(N)}{g_\gamma(N_0)} \right] = \mu' (t - t_0) . \qquad (18)$$

where we used the notation $g_\gamma(x) \equiv e^{-\gamma x} - 1$. This equation can be numerically solved and the result shown in figure 4a for a given set of parameters that was chosen to provide a similar range of time and N values than the original series. We can appreciate from this diagram that there is a delayed growth phase at the beginning followed by an apparently linear growth in late stages. In other words, the dynamics has no singularity. Is that the case? Although solving the general problem can be extremely cumbersome, we can deal with some approximations that can be applied at different stages of the system.

The first considers the case $\gamma << 1$ and the initial phase of the expansion. If N is not large then $\gamma N << 1$ and we can use the approximation $e^{-\gamma N} \simeq 1 - \gamma N$. on
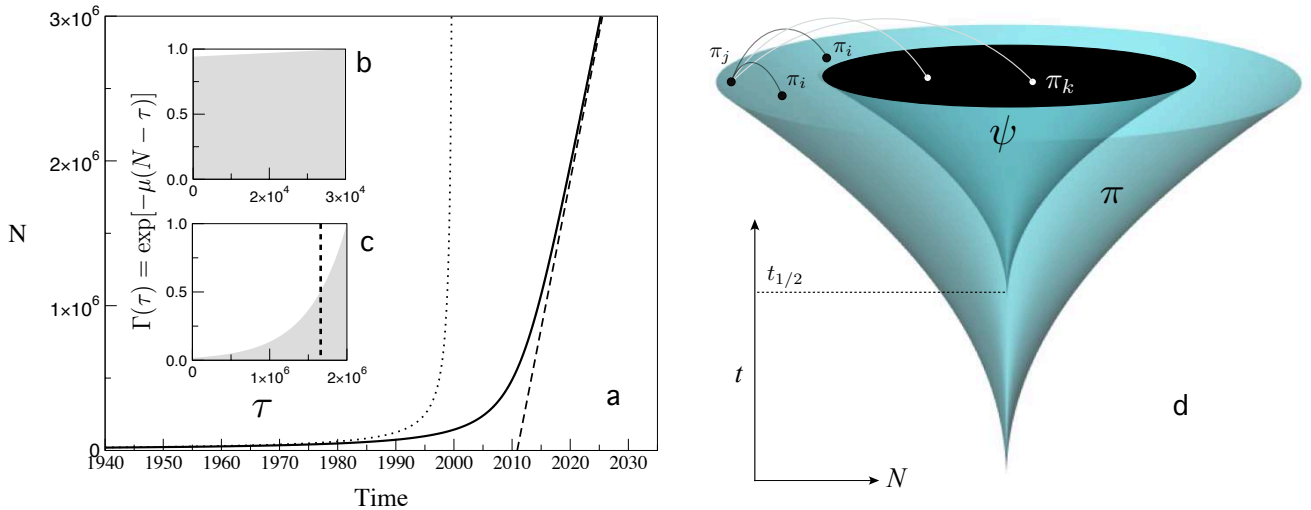
Figure 4. Transient hyperbolic growth and blackholes in combination models with limited memory. In (a) we display our predicted growth curve $N(t)$ and two approximations considering short time (dotted line) and long term (dashed line) scales. The effective kernels for these two scales are displayed in the inset plots (b) and (c). The maximum value displayed in the τ axis of insets (b) and (c) corresponds to $N(t)$ at times $t = 1963$ and $t = 2020$, respectively. The characteristic scale of ageing imposed by the kernel implies that there is a time horizon beyond which no connections among inventions can be made. This illustrated schematically in (d) where we shown the spindle diagram of the whole system π along with a subset $\psi(t)$, first appearing at the characteristic time $t_{1/2}$ where the probability of citing the oldest invention π_1 is half the maximum. All inventions within this "black hole" will be disconnected from the rest. In the present (top large circle) only new inventions (filled circles) occupying the outer part of the circle can connect among them whereas they cannot link (light lines) with those in the black hole (open circles). The parameter values used in (a)-(c) correspond to $\mu = 2 \times 10^5$, $t_0 = 1750$, $N_0 = 5000$, and $\gamma = 2 \times 10^{-6}$.

Eq. (19) leading to:

$$N - N_0 - \frac{1}{\gamma^2}\left[\frac{1}{N} - \frac{1}{N_0}\right] +$$
$$+ \frac{1}{\gamma}\ln\left[\frac{N}{N_0}\right] = \mu'\,(t - t_0)\,. \qquad (19)$$

Given the condition $\gamma << 1$, the LHS of the previous equation is governed by the quadratic terms of γ. Hence, neglecting the first and last terms on the LHS, we obtain the following approximate solution for the evolution of N at initial stages of technological evolution:

$$N(t) = \frac{1}{\gamma^2\mu'\,(t_s - t)} \qquad (20)$$

where $t_s = 1/(\mu'\gamma^2 N_0) + t_0$. The previous equation predicts a singularity at some time in the future although such singularity is in conflict with our approximation and the hyperbolic growth is only a transient phenomenon.

Let us know focus on the long-term $(t \to \infty)$ dynamics of the system. In this case it is sensible to assume a very large number of existing patents, and hence $e^{-\gamma N} \to 0$. In this case, we can rewritte Eq. (19) as:

$$(N - N_0) - \frac{1}{\gamma} + \frac{1}{\gamma^2 N_0} - \frac{1}{\gamma}\ln\left[\gamma N_0\right] = \mu'\,(t - t_0)\,, \qquad (21)$$

where we have also used again $e^{-\gamma N_0} \simeq 1 - \gamma N_0$. Thus, from Eq. (23) it is straightforward to obtain the long-term solution for the dynamics of N:

$$N(t) = \mu'\,(t - t_0) + \varepsilon \qquad (22)$$

with $\varepsilon = N_0 + 1/\gamma - 1/(\gamma^2 N_0) + \ln\left[\gamma N_0\right]/\gamma$.

Equation (24) reveals that N exhibits a linear growth dynamics when large values of t are considered. Note that this long-term dynamics is notably different from the hyperbolic dynamics predicted for initial stages of evolution (see the explanation above). In figure 4 we show the fits of these approximations to the exact solution (obtained numerically). As we can see, the analytic results confirm that the initial hyperbolic trend (dotted curve)is eventually replaced by a slowdown characterised by a linear process (dashed line) with no technological singularity associated.

The previous results can be generalised to the k-diversity scenario, where the new equation reads

$$\frac{dN}{dt} = (-1)^{k+1}\mu\prod_{l=1}^{k}\left[\int_0^N e^{-\gamma(N-\tau_1)}d\tau_1\right] \qquad (23)$$

where the term $(-1)^{k+1}$ is required in order to avoid the unphysical situation in which the number of patents decreases when combining an odd number of previous inventions (note that a negative integration constant is obtained when integrating e^{-x}).

This general model leads to:

$$\int_{N_0}^{N}\frac{d\widetilde{N}}{\left[e^{-\gamma\widetilde{N}} - 1\right]^k} = \mu'\int_{t_0}^{t}d\widetilde{t} \qquad (24)$$

which, at initial stages reads:

$$\int_{N_0}^{N}\frac{d\widetilde{N}}{\left(\gamma\widetilde{N}\right)^k} = \mu'\int_{t_0}^{t}d\widetilde{t}, \qquad (25)$$

and gives hyperbolic dynamics, whereas in the long-term dynamics we have now:

$$\int_{N_0}^{N} d\widetilde{N} = \mu' \int_{t_0}^{t} d\widetilde{t}, \qquad (26)$$

again leading to linear dynamics.

An intuitive explanation for the change from early hyperbolic to late linear dynamics is provided in the insets of figure 4b-c. Here we show the kernel associated to early (fig 4b) and late (fig 4c) times, and thus smaller and larger numbers of innovations. Although the area covered by Γ almost fills the plot when $N \sim O(10^4)$, it becomes smaller with large N values (here $N \sim O(10^6)$). In Fig. 4b, the probability of recombining any existing patent is higher than 90% of the maximum [i.e., 90% of the probability of recombining the newest pattent $N(t)$, with $t = 1963$]. However, the probability of combining patents existing at $t = 2020$ (Fig. 4c) folds to approximately zero for the oldest patents. Then, we can arbitrarily define a patent number $\pi_h(t)$ that delimits the frontier between up-to-date patents and obsolete patents (which will hardly ever been recombined again). Specifically, we consider $\pi_h(t)$ to be the patent number for which the probability of recombination is half the maximum (as indicated by the dashed line in Fig. 4c). Thus, the effect of ageing (or loss of memory) dominates in the long term (Figs. 4a and c) and the effective rate of innovation become linear. Such slowdown prevents the system from approaching a divergent dynamics.

These results can be graphically interpreted as shown in figure 4d. Here we use again the spindle diagram showing how the universe (or space) of innovations experiences an accelerated growth at early stages of development. Novel patents such as π_j will be distributed over the outer parts of the patent space (a circle at each time step) and connect with others such as π_i. After a critical time $t_{1/2}$, some inventions start to become obsolete or forgotten. From this time on, the expansion speed stabilises, and both the universe of inventions and the "blackhole" $\psi(t)$ at its center (which represents the area of obsolete technology) grow at the same constant speed. Inventions within the blackhole (such as π_k in figure 4d) cannot be used and thus no information about them can cross the obsolescence frontier. Our technological memory establishes the distance between these two boundaries in the innovations space, and this distance determines the number of up-to-date inventions, which in turn determine the expansion rate of the innovation space.

4. Discussion

The nature and tempo of innovation is a difficult and timely topic. It has been the focus of attention from evolutionary biologists, economists and physicists alike. Inventors get inspiration from previous, existing designs, while they push forward the boundaries of innovation. In searching for a theory of technological change, the combinatorial nature of technological innovation seems to be an essential component of human creativity. By combining previous designs into novel inventions, there is a potential for an explosion of novelties, which

could eventually move towards a singularity. How can we test such possibility? Patent citations are a privileged window into such process, since they provide a first approximation to both the growth of inventions and their interactions over time. The accelerated pattern of patent growth suggests that a super linear process of innovation is taking place and available evidence indicates that this is at least partially associated to combinatorial processes (18).

In this paper we have explored a simple class of models that include both the richness of combinations and how rapidly the relevance of previous inventions fades with time. These two features can be seen as two opposing forces: the diversity of potential previous inventions to be combined powers combinatorial design, while the obsolescence of the same inventions makes them less likely to contribute to combinations. Our goal was not as much as to fit data than understand the basic scenarios where singularities might emerge when both features are included.

We have shown that long-memory kernels permit the presence of singularities under some conditions, while kernels involving a characteristic time scale of ageing forbid divergences to occur. The first class predicts two different phases, which reminds us of a picture of innovation defining a phase transition between sub-critical and super-critical phases (29). The second provides a plausible explanation of why singularities might fail to be observed, while the transient dynamics of innovation appears hyperbolic. Further investigations should analyse other temporal trends (including the patterns of fluctuations) associated to these class of models and a more details analysis of available time series. Existing models of evolution of innovations (30,31) can provide very useful tests to the ideas outlined here.

Acknowledgments

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