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Abstract. Connectedness is a fundamental property of objects and systems. It is usually viewed as inherently topological, and hence treated as derived property of sets in (generalized) topological spaces. There have been several independent attempts, however, to axiomatize connectedness either directly or in the context of axiom systems describing separation. In this review-like contribution we attempt to link these theories together. We find that despite difference in formalism and language they are largely equivalent. Taken together the available literate provides a coherent mathematical framework that is not only interesting in its own right but may also be of use in several areas of computer science from image analysis to combinatorial optimization.

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1. Introduction

Connectedness is a fundamental property of objects and thus plays a key role in particular in devising computational models for them. In topology it was studied already in the early 1900s by Hausdorff, Riesz, and Lennes (see [1] for a historic perspective on this most early work). Topological spaces (and their generalizations known as closure spaces) come endowed with a natural concept of topological connectedness that is usually expressed in terms of a separation relation: two sets A and B are separated if $(A \cap c(B)) \cup (c(A) \cap B) = \emptyset$. The Hausdorff-Lennes condition stipulates that a set is connected if it cannot be partitioned into two non-empty separated subsets. Connectedness thus is usually treated as a derived property of spaces that are defined in terms of notions of boundary, closure, or interior. An early exposition of connectedness is [2]. From a category theory point

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of view, connectedness is instead defined in terms of continuity, using a classical theorem from topology as definition: X is ζ-connected if and only if every ζ-continuous function $f : X \to \nvDash$ is constant, where $\nvDash$ is the discrete space with two points [3, 4, 5].

Generalized topologies $(X, \mathcal{O})$ in the sense of Császár [6] consist of a set system $\mathcal{O} \subseteq 2^X$ that is closed under arbitrary unions, i.e., if $O_\iota \in \mathcal{O}$ for $\iota \in I$, then $\bigcup_{\iota \in I} O_i \subseteq \mathcal{O}$. The elements of $\mathcal{O}$ serve as generalization of open sets. In this setting a space X is $\mathcal{O}$-connected if there are no two disjoint non-empty open sets $O_1, O_2 \in \mathcal{O}$ so that $O_1 \cup O_2 = X$. A variety of derived generalized open set systems can be constructed in terms of the closure and interior operators on $\mathcal{O}$, thus giving rise to different flavors of connectedness, see e.g. [7, 8, 9] for a systematic analysis.

In the 1940s, several authors investigated connectedness as the fundamental concept of a topological theory starting from axioms for separation between sets instead of deriving separation from another topological structure [10, 11, 12]. It was soon recognized this leads to theories that are substantially more general than topological spaces [12, 13, 14]. In particular, there are natural connectivity structures that do not coincide with the connected sets of any topological space (or generalized closure space). A well-known example are the arc-wise connected sets in the plane [15]. The notion of separation is intimately related to that of *proximity* [16]. A set A is proximally-connected in a proximity space (X, δ) if, for any two non-empty sets A', A'' with $A' \cup A'' = A$ holds $A' \delta A''$ [3], i.e., if A cannot be separated into a pair of far sets, see [17] for some further developments.

Starting with the 1980s, several authors have begun again to investigate systems of connected sets either in their own right [18, 19] or motivated by particular applications. A close connection between knot theory and finite connectivity spaces has been explored in [20]. Connectivity is also an important concept in digital image analysis and has become a focus in a mathematical morphology starting with the work of Serra, Ronse, and collaborators [21, 22] and centers on filter operations, called openings or closings, that remove grains or fill in pores [23] and provide an abstract definition of connected components. Several natural constructions in this context lead to collections of "connected sets" that are not derivable from topologies. More recently, the investigation of generalized topologies associated with chemical reaction networks has lead to "constructive connectedness" as a more natural notion of connectivity of chemical spaces than the Hausdorff-Lennes connectivity on the same neighborhood space [24].

An axiomatic approach to connectivity is also of interest in the context of fitness landscapes, i.e., functions $f : X \to \mathbf{R}$, where $\mathbf{R}$ is some well-ordered set, X is the underlying search space, and f is a fitness or cost function. While X is finite (but usually unmanageably large) in combinatorial optimization problems, X is usually taken to be a continuum in the field of evolutionary computation. In both settings, coarse grained representations (such as barrier trees [25]), as well as elaborate stochastic models of optimization algorithms make prominent use of the connected components of "level sets" $F_h = \{x | f(x) \leq h\}$. A closer inspection shows that the only intrinsic structure of the search space X that is actually used in this context is the connectedness of its subsets. This is most transparent in Trouvé's "cycle decomposition" of the state space [26] in the theory of simulated annealing. The connectedness of the set of optimal solutions in a multi-objective optimization problem, furthermore, has an impact on the performance of heuristics [27]. Minimax theorems for functions of the form $g : X \times Y \to \mathbb{R}$ also involve the connectedness of level sets in one variable. This generates a separation structure in the sense of Wallace rather than a topology on the sets X and Y [28].

Connectedness, and in particular the notion of connected components, are of key interest in topological approaches to image analysis. It is of imminent practical importance for image filtering and segmentation, image compression and coding, motion analysis, and pattern recognition, see e.g. [29, 30, 31, 32].

It is the purpose of this contribution to summarize elementary results on connectivity spaces and their associated separation relations. Much of the material compiled here is "mathematical folklore" and most results have already been obtained by others in one form or the other. Since we strive to present the connections between the different formalisms in their most general form, we nevertheless include simple proofs of many statements that have been published only in a less general setting or for which we could not find an easily accessible proof in the literature. We deliberately concentrate on simple, very basic properties of connectivity structure in an attempt to connect independent lines of reasoning and results that have been scattered in the literature.

2. General Connectivity Spaces

2.1. Basic Definitions

Throughout the contribution we are interested in connectedness structures on an arbitrary set X. The most direct approach is to specify axioms for a collection $\mathcal{C} \subseteq 2^X$ of *connected subsets* of X [21, 18, 19, 20].

Definition 2.1. A connectivity space is a pair $(X, \mathcal{C})$ with $\mathcal{C} \subseteq 2^X$ so that

(c0) $\emptyset \in \mathcal{C}$

(c1) $Z_\imath \in \mathcal{C}$ for all $\imath \in I$ and $\bigcap_{\imath \in I} Z_\imath \neq \emptyset$ implies $\bigcup_{\imath \in I} Z_\imath \in \mathcal{C}$

Connectivity spaces have also been termed "connective spaces" or "c-spaces".

Consider an arbitrary set $Z \in \mathcal{C}$ and let $\mathcal{B}_Z$ be a subset of $\mathcal{C}$ so that (i) $Z \in \mathcal{B}_Z$ and (ii) $Z' \in \mathcal{B}_Z$ implies $Z' \cap Z \neq \emptyset$.

Fact 2.2. *$\mathcal{C}$ satisfies axiom (c1) if and only if the following holds:*

(c1') *Let $Z \in \mathcal{C}$ and $\mathcal{B}_Z \subseteq \mathcal{C}$ so that (i) $Z \in \mathcal{B}_Z$ and (ii) $Z' \in \mathcal{B}_Z$ implies $Z' \cap Z \neq \emptyset$. Then $\bigcup_{Z' \in \mathcal{B}_Z} Z' \in \mathcal{C}$.*

Proof. Suppose (c1) holds and Z, Z' and $\mathcal{B}_Z$ is defined as in (c1'). Then $Z' \cup Z \in \mathcal{C}$ for all $Z' \in \mathcal{B}_Z$. Furthermore $\bigcup_{Z' \in \mathcal{B}_Z} Z' = \bigcup_{Z' \in \mathcal{B}_Z} (Z' \cup Z)$. The latter sets are connected and their intersection contains Z, hence their union is connected as well by axiom (c1). The converse is obvious. □

2.2. Connected Components

The concept of connected components is a key ingredient of any theory of connectivity.

Definition 2.3. For every $A \subseteq X$ and every $x \in X$ the set

$$A[x] = \bigcup \{ A' \subseteq X \mid A' \subseteq A,\, x \in A',\, A' \in \mathcal{C} \} \tag{2.1}$$

is called the *connected component* of $x \in A$.

By definition $A[x] = \emptyset$ if $x \notin A$. Furthermore $A[x] \in \mathcal{C}$ as a direct consequence of axiom (c1) for non-empty $A[x]$ and of axiom (c0). It is important to note that $x \in A$ does not guarantee that $A[x]$ is non-empty.

The following observation has been made repeatedly in the literature albeit for more restricted types of connectivity spaces.

Fact 2.4. *Given an arbitrary collection $\mathcal{B} \subseteq 2^X$ of connected sets there is a unique minimal collection $\mu'(\mathcal{B}) \subseteq 2^X$ such that $\mathcal{B} \subseteq \mu'(\mathcal{B})$ and $\mu'(\mathcal{B})$ satisfies (c0) and (c1).*

Proof. Define $\mu'(\mathcal{B})$ as the intersection of all systems of all connectivity spaces $(X, \mathcal{C})$ with $\mathcal{B} \subseteq \mathcal{C}$. It is non-empty since $(X, 2^X)$ is a connectivity space. Consider a collection $Z_\imath \in \mu'(\mathcal{B})$, $\imath \in I$. By construction $Z_\imath \in \mu'(\mathcal{B})$ implies $Z_\imath \in \mathcal{C}$ for all $\mathcal{C}$ containing $\mathcal{B}$ and by (c1) we have $\bigcup_{\imath \in I} Z_i \in \mathcal{C}$ for each $\mathcal{C}$ containing $\mathcal{B}$. Thus (c1) is satisfied on $\mu'(\mathcal{B})$. Axiom (c0) holds trivially. □

This argument can be found e.g. in [19] in a slightly more restrictive context. By construction $\mu' : 2^{2^X} \to 2^{2^X}$ is idempotent, i.e., $\mu'(\mu'(\mathcal{B})) = \mu'(\mathcal{B})$ and expanding, i.e., $\mathcal{B} \subseteq \mu'(\mathcal{B})$. We call $\mathcal{B}$ a basis of the connectivity space $\mu'(\mathcal{B})$. Furthermore, we say that $\mathcal{B}$ is *complete* if $\mathcal{B} = \mu'(\mathcal{B})$, i.e., if the basis already satisfies (c0) and (c1).

It is also possible to obtain the connected components w.r.t. $\mu'(\mathcal{B})$ directly from the basis $\mathcal{B}$ using a construction first described in [33]:

For given $A \in 2^X$ and $x \in A$ let $A_0[x] = \bigcup\{Z \in \mathcal{B} | Z \subseteq A,\, x \in Z\}$. Then define recursively $A_k[x] = A_{k-1}[x] \cup \bigcup\{Z \in \mathcal{B} | Z \subseteq A,\, Z \cap A_{k-1}[x] \neq \emptyset\}$ and set $A^*[x] = \bigcup_{k=0}^{\infty} A_k[x]$. $A_k[x] \in \mu'(\mathcal{B})$ by Fact 2.2, $A^*[x]$ exists since $A_{k-1}[x] \subseteq A_k[x] \subseteq A$, and $A^*[x] \in \mu'(\mathcal{B})$ since $(X, \mu'(\mathcal{B}))$ satisfies (c1). Furthermore, $A^*[x] = A$ whenever $A \in \mathcal{B}$. Hence the system of connected sets $\{A^*[x] | A \in 2^X,\, x \in A\}$ contains $\mathcal{B}$, satisfies (c0) and (c1) and is contained in $\mu'(\mathcal{B})$. Thus it coincides with $\mu'(\mathcal{B})$. In particular, $A^*[x] = A[x]$, the connected components w.r.t. $\mu'(\mathcal{B})$. Note that $A[x]$ can be empty since $A^*[x] = \emptyset$ if and only if $A_0[x] = \emptyset$. The same result is obtained in [34, Thm.3] using transfinite induction.

In this most general setting, therefore, the set system $\{A[x] | x \in A\}$ does not define a partition of A and may even consist of the empty set only.

2.3. Connectivity Openings

An alternative starting point to the theory of connectivity spaces are the properties of connected components. This avenue was explored e.g. in [21, 22, 35].

Definition 2.5. A "connectivity opening" is a map $\gamma : X \times 2^X \to 2^X : (x, A) \mapsto A[x]$ that satisfies for all $x \in X$ and all $A, B \in 2^X$ the following axioms:

(o0) $x \notin A$ implies $A[x] = \emptyset$.
(o1) $A[x] \subseteq A$.
(o2) $A \subseteq B$ implies $A[x] \subseteq B[x]$.
(o3) $(A[x])[x] = A[x]$.
(o4) $A[x] \cap A[y] = \emptyset$ or $A[x] = A[y]$.

For later reference we note (o2) is equivalent to

$$A[x] \cup B[x] \subseteq (A \cup B)[x] \tag{2.2}$$

The following result slightly generalizes [21, Thm. 2.8].

Theorem 2.6. *There is one-to-one correspondence between systems of connected components satisfying axioms (o0) to (o4) and connectivity spaces satisfying (c0) and (c1) given by equ.(*2.1*) and*

$$\mathcal{C} = \left\{ Z \in 2^X \middle| Z = A[x],\ A \subseteq X,\ x \in X \right\} \tag{2.3}$$

Proof. Suppose $(X, \mathcal{C})$ satisfies (c0) and (c1). Properties (o0), (o1), and (o2) follow immediately from equ.(2.1). (o3) follows from $A[x] \in \mathcal{C}$, (o1), and equ.(2.1). To show (o4) we assume that $A[x] \cap A[y] \neq \emptyset$. Then $A[x] \cup A[y] \in \mathcal{C}$ by (c1), and hence, by (o0) $x \in A[y]$ and $y \in A[y]$. Equ.(2.1) thus implies $A[y] \subseteq A[x]$ and $A[x] \subseteq A[y]$, i.e., $A[x] = A[y]$.

Conversely, consider a connectivity opening. Then the set system $\mathcal{C}$ defined by equ.(2.3) satisfies (c0) as a consequence of (o1). To see that (c1) holds fix some $z \in X$ and a subset $\mathcal{B} \subseteq \mathcal{C}$ so that $z \in C = C[z]$ for all $C \in \mathcal{B}$. We compute

$$\bigcup_{C \in \mathcal{B}} C = \bigcup_{C \in \mathcal{B}} C[z] = \bigcup_{C \in \mathcal{B}} (C[z])[z] \subseteq \left(\bigcup_{C \in \mathcal{B}} C[z] \right)[z] \subseteq \left(\bigcup_{C \in \mathcal{B}} C \right)[z] \subseteq \bigcup_{C \in \mathcal{B}} C$$

by using $C = C[z]$, axiom (o3), equ.(2.2), $C = C[z]$, and axiom (o1). By equ.(2.3) we have $\left(\bigcup_{C \in \mathcal{B}} C\right)[z] \in \mathcal{C}$ and thus (c1) holds. □

2.4. Separation Relations

Attempts to capture connectedness in terms of separations of sets go back to the first half of the 20th century and are intimately related to the foundations of topological structures.

Definition 2.7. A *symmetric separation* on X is a collection $\mathfrak{S} \subset 2^X \times 2^X$ satisfying the axioms

(S0) $(A, \emptyset) \in \mathfrak{S}$
(S1) $(X, Y) \in \mathfrak{S}$, $X' \subseteq X$, and $Y' \subseteq Y$ implies $(X', Y') \in \mathfrak{S}$.
(S2) $(X, Y) \in \mathfrak{S}$ implies $(Y, X) \in \mathfrak{S}$

A system of pairs $\mathfrak{S}$ is called *grounded* if it satisfies (S0), *hereditary* if (S1) is satisfied, and *symmetric* if (S2) holds. The pair $(X, \mathfrak{S})$ is sometimes called a separation space.

The symmetric separations defined here are even more general than the ones studies by Wallace [12] and Hammer [13, 14, 33]. Non-symmetric separation have been considered as an alternative basis of topology and in the context of quasi-proximities [36]. Since connectivity is inherently symmetric, however, there is nothing to be gained for our purposes by dropping axiom (S2). We note that separations are equivalent to so-called semi-topogenous orders on 2^X by virtue of $(A, B) \in \mathfrak{S}$ iff $A \prec X \setminus B$ [37, 38]. The results outlined in this section are mostly generalizations of the early literature on separation spaces [12, 13, 38].

Definition 2.8. Let $\mathfrak{S}$ be a symmetric separation on X. A set $Z \subseteq X$ is $\mathfrak{S}$-*connected* if there is no pair $(A, B) \in \mathfrak{S}$ so that $Z \neq A$, $Z \neq B$, and $A \cup B = Z$.

Lemma 2.9. *Equivalently, Z is $\mathfrak{S}$-connected if, for every $(A, B) \in \mathfrak{S}$ with $Z \subseteq A \cup B$ we have $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$.*

Proof. Since $\mathfrak{S}$ is hereditary, $(A, B) \in \mathfrak{S}$ and $Z \subseteq A \cup B$ implies $(A \cap Z, B \cap Z) \in \mathfrak{S}$ and $(Z \cap A) \cup (Z \cap B) = Z$. If one of $(Z \cap A)$ or $(Z \cap B) = Z$ is empty, then $Z = A$ or $Z = B$, i.e., Z is $\mathfrak{S}$-connected. Thus the condition is sufficient. To show that it is also necessary, suppose for all $(A, B) \in \mathfrak{S}$ with $A \cup B = Z$ we have either $Z = A$ or $Z = B$, i.e., Z is $\mathfrak{S}$-connected, but neither A nor B is empty. Thus $A \cap B \neq \emptyset$. By symmetry of $\mathfrak{S}$ we may assume w.l.o.g. that $A = Z$. If $B \neq A$, $(Z \setminus B, B) \in \mathfrak{S}$, satisfies $(Z \setminus B) \cup B = Z$ and neither B nor $Z \setminus B$ equals Z, thus Z is separated by $(Z \setminus B, B) \in \mathfrak{S}$, contradicting $\mathfrak{S}$-connectedness of Z. If $A = B = Z$ we can split A into an arbitrary pair of disjoint non-empty sets and arrive at the same contradiction. Thus either A or B must be empty. □

Theorem 2.10. *The collection $\mathcal{C}_{\mathfrak{S}}$ of $\mathfrak{S}$-connected sets on X satisfies (c0) and (c1).*

Proof. By construction $\emptyset \in \mathcal{C}_{\mathfrak{S}}$, i.e., (c0) holds. Now suppose $z \in \bigcap_{Z\in\mathcal{B}} Z$ for some collection $\mathcal{B} \subseteq \mathcal{C}_{\mathfrak{S}}$. Fix $(A, B) \in \mathfrak{S}$ and consider a collection $\mathcal{B} \subseteq \mathcal{C}_{\mathfrak{S}}$ of $\mathfrak{S}$-connected sets so that (i) $Z \subseteq A \cup B$ and (ii) there is $z \in Z$ for all $Z \in \mathcal{B}$. W.l.o.g. we can assume that $x \in A$. Lemma 2.9 thus implies $Z \cap B = \emptyset$ for all $Z \in \mathcal{B}$. Thus

$$\bigcup_{Z\in\mathcal{B}} (Z \cap B) = \left(\bigcup_{Z\in\mathcal{B}} Z \right) \cap B = \emptyset$$

while $\bigcup_{Z\in\mathcal{B}} Z \subseteq A \cup B$. Thus $\bigcup_{Z\in\mathcal{B}} Z$ is $\mathfrak{S} - connected$ and hence contained in $\mathcal{C}_{\mathfrak{S}}$. □

Note that there can be more than one separation that generates the same collection of $\mathfrak{S}$-connected set. Following [12] we see that union $\mathfrak{S}_1 \cup \mathfrak{S}_2$ of two separations with $\mathcal{C}_{\mathfrak{S}_1} = \mathcal{C}_{\mathfrak{S}_2}$ is again a symmetric separation and generates the same connected sets. Thus there is a unique maximal symmetric separation with this property.

Conversely, we may start from an arbitrary collection $\mathcal{B}$ of subsets of X and construct the finest separation consistent with $\mathcal{B}$ as

Definition 2.11.

$\mathfrak{S}_{\mathcal{B}} := \big\{(A,B) \in X \times X \big| \forall Z \in \mathcal{B} \text{ with } Z \subseteq A \cup B \text{ holds} Z \cap A = \emptyset \text{ or } Z \cap B = \emptyset\big\}$

Lemma 2.12. $\mathfrak{S}_{\mathcal{B}}$ *satisfies (S0), (S1), and (S2) for every set* $\mathcal{B} \in 2^X$.

Proof. Axiom (S0) and symmetry (S2) follow directly from the definition. Suppose $Z \subseteq A \cup B$ implies $Z \cap A = \emptyset$ or $Z \cap B = \emptyset$. Then $Z \cap A' = \emptyset$ or $Z \cap B' = \emptyset$ for all $A' \subseteq A$ and $B' \subseteq B$ and thus in particular for all (A', B') that still satisfy $Z \subseteq A' \cup B'$. Thus $\mathfrak{S}_{\mathcal{C}}$ is hereditary. □

Taken together, for every separation $\mathfrak{S}$ there is a finest separation $\widehat{\mathfrak{S}}$ that generates the same (complete) collection of connected sets $\mathcal{C}$. We have $\mathfrak{S} \subseteq \widehat{\mathfrak{S}}$. By construction $\mathcal{C}_{\mathfrak{S}_{\mathcal{B}}}$ is the smallest collection of connected sets that contains $\mathcal{B}$ and satisfies (c0) and (c1), whence

$$\mu'(\mathcal{B}) = \mathcal{C}_{\widehat{\mathfrak{S}_{\mathcal{B}}}} = \mathcal{C}_{\mathfrak{S}_{\mathcal{B}}} \tag{2.4}$$

Finally, the following direct characterization of a complete collection of connected sets may be useful.

Lemma 2.13. $Z \in \mu'(\mathcal{B})$ *if and only if, for every non-empty proper subset* $A \subset Z$ *there is* $C \in \mathcal{B}$ *so that* $A \cap C \neq \emptyset$ *and* $(Z \setminus A) \cap C \neq \emptyset$.

Proof. Z is $\mathfrak{S}_{\mathcal{B}}$-connected iff, for all non-empty proper $A \subset Z$ we have $(A, Z \setminus A) \notin \mathfrak{S}_{\mathcal{B}}$, i.e., iff for every A there is a $C_A \in \mathcal{B}$ so that $(C_A \cap A, (Z \setminus A) \cap C_A) \notin \mathfrak{S}_{\mathcal{B}}$. The latter condition is equivalent to requiring C_A intersects both A and $Z \setminus A$. □

In the following we will always assume that $\mathcal{C}$ satisfies (c0), (c1) and is complete and that $\mathfrak{S}$ is the finest symmetric separation for a given complete connectivity.

For later reference we note the following equivalence, which is the negated version of Def. 2.11.

Fact 2.14. $(A, B) \notin \mathfrak{S}$ *if and only if there is* $U \in \mathcal{C}$, $U \subseteq A \cup B$, *so that* $A \cap U \neq \emptyset$ *and* $B \cap U \neq \emptyset$.

As an immediate corollary we have

Fact 2.15. *If* $(X, \mathcal{C})$ *satisfies (c1) and* $Z', Z'' \in \mathcal{C}$. *Then either* $Z' \cup Z'' \in \mathcal{C}$ *or* $(Z', Z'') \in \mathfrak{S}_{\mathcal{C}}$.

Theorem 2.16. *For every* $A \subseteq X$ *and* $x \in A$ *we have* $(A[x], A \setminus A[x]) \in \mathfrak{S}$.

Proof. By definition of the connected component $A[x]$ there is no connected set Z that intersects both $A[x]$ and $A \setminus A[x]$. Hence Fact 2.14 implies that $A[x]$ and $A \setminus A[x]$ are separated. □

Finally, we note that singletons, i.e., sets containing a single point, need not be connected in general. Indeed, a point $x \in X$ is disconnected if there is a pair $(A, B) \in \mathfrak{S}$ so that $x \in A \cap B$. In particular, every set $A \subseteq X$ can be partitioned into its connected components and a set of disconnected points $A^{\cup}$ characterized by $(\{x\}, \{x\}) \in \mathfrak{S}$. Note that $x \in A^{\cup}$ may be contained in a connected component $B[x]$ for some set B with $A \subset B$.

2.5. Subspaces

A subset $Y \subseteq X$ inherits the connectivity structure $\mathcal{C}$ of X by means of $\mathcal{C}_Y = \{A \in \mathcal{C} | A \subseteq Y\}$. Analogously, the separation relation $\mathfrak{S}$ is inherited by means of $(A, B) \in \mathfrak{S}_Y$ if and only if $(A, B) \in \mathfrak{S}$ and $A, B \subseteq Y$.

2.6. Separations and Isotonic Closure Spaces

The connection between separations and isotonic closure spaces was investigated in the mid 20th century [12, 14, 15] in the context of Wallace separations and even more restrictive variants of proximity spaces [39]. In 2005, Harris [40] considered the general case of symmetric separations satisfying only (S1) and (S2).

Topological spaces and their generalization can be characterized by Kuratowski's axioms for a closure function $c : 2^X \to 2^X$:

(K0) $c(\emptyset) = \emptyset$. (grounded)
(K1) $A' \subseteq A$ implies $c(A') \subseteq c(A)$ for all $A \subseteq X$. (isotone)
(K2) $A \subseteq c(A)$ for all $A \subseteq X$. (enlarging)
(K3) $c(A \cup B) \subseteq c(A) \cup c(B)$ for all $A, B \subseteq X$. (subadditive)
(K4) $c(c(A)) = c(A)$ for all $A \subseteq X$. (itempotent)

A closure space is point-wise symmetric if it satisfies

(R0) $x \in c(\{y\})$ implies $y \in c(\{x\})$ for all $x, y \in X$.

For a closure space (X, c) one can define an associated separation in terms of the so-called Hausdorff-Lennes condition:

$$\mathfrak{S}_{HL} = \left\{(A, B) \in 2^X \times 2^X \middle| (A \cap c(B)) \cup (c(A) \cap B) = \emptyset\right\} \tag{2.5}$$

Topologically connected sets are those that are $\mathfrak{S}_{HL}$-connected.

Starting from an arbitrary relation $\mathfrak{S} \subseteq 2^X \times 2^X$ we define, following [14], the *Wallace function* of $\mathfrak{S} \subseteq 2^X \times 2^X$ by

$$w(A) = \bigcap \{B \subseteq X | (A, X \setminus B) \in \mathfrak{S}\} \tag{2.6}$$

We can regard the Wallace function w of $\mathfrak{S}$ as a generalized closure operator on X. As shown in [13], the Wallace function has a simpler representation as

$$w(A) = \{x \in X | (\{x\}, A) \notin \mathfrak{S}\}$$

for every $A \subseteq X$. Using Fact 2.14 this can be rephrased as $x \in w(A)$ if any only if there is a connected set $Z \subseteq A \cup \{x\}$ so that $Z \cap A \neq \emptyset$. Thus the elements of $w(A)$ are exactly the *touching points* of [19], see also [14, Thm.4.9].

If $\mathfrak{S}$ is hereditary (S1), then $w : 2^X \to 2^X$ is isotonic (K1). Furthermore (S0) implies $(\emptyset, X) \in \mathfrak{S}$ and hence $w(\emptyset) = \emptyset$, i.e., (K0). Since by definition $w(\{x\}) =$

$\{y \in X | (\{x\}, \{y\}) \notin \mathfrak{S}\}$, we have $y \in w(\{x\})$ if and only if $(\{x\}, \{y\}) \notin \mathfrak{S}$, i.e., if and only if $(\{y\}, \{x\}) \notin \mathfrak{S}$, i.e., iff $x \in w(\{y\})$. Thus w satisfies the symmetry axiom (R0).

Not all symmetric separations derive from generalized closure functions, however. An additional condition, used e.g. in [12, 41, 36] in the context of (quasi)proximities is required:

(SX) If $(\{x\}, B) \in \mathfrak{S}$ for all $x \in A$ and $(A, \{y\}) \in \mathfrak{S}$ for all $y \in B$ then $(A, B) \in \mathfrak{S}$

The following result is a variant of (3.3.) in [12], Theorem 3.2 of [14], see also [36] and [40, Thm.3]:

Theorem 2.17. *If $\mathfrak{S}$ satisfies (S1) and (S2) and (SX) then $(A, B) \in \mathfrak{S}$ if and only if $A \cap w(B) = \emptyset$ and $w(A) \cap B = \emptyset$, i.e., $\mathfrak{S}$ is the Hausdorff-Lennes separation of the pointwise symmetric isotone closure space (X, w).*

The properties of point-symmetric closure spaces are characterized by their Hausdorff-Lennes separations, since the separations uniquely determine the corresponding closure.

Fact 2.18. *If $\mathfrak{S}$ satisfies (S2) and (SX), i.e., if the corresponding closure function w is pointwise symmetric (R0), then $\mathfrak{S}$ is grounded if and only if (X, w) is grounded (K0).*

Proof. In [40] the equivalence of (K0) with the (in general weaker) axiom "$(\{x\}, \emptyset) \in \mathfrak{S}$ for all $x \in X$" is shown. This is equivalent to (S0), however, if $\mathfrak{S}$ is symmetric (S2) and satisfies (SX). □

In isotone closure spaces some of the theorems regarding connected sets that are well known from topological spaces still hold or at least have simple generalizations [42]:

1. If $f : (X, c) \to (Y, c')$ is continuous and X is connected, then Y is connected.
2. (X, c) is connected if and only if all continuous functions $Y \to X$ are constant for every isotone space $(\{x, y\}, c)$ with $c(\{x\}) = \{x\}$ and $c(\{y\}) = \{y\}$.
3. If A is connected and $A \subseteq c(A)$, then $c(A)$ is connected.

Following Thm.1.4. of [7] we may also conclude that B is connected if $A \subseteq B \subseteq c(A)$ and A is connected.

We conclude this section by expressing axiom (SX) in terms of connected sets:

(cX) For all $Z \in \mathfrak{C}$ and all $A \neq \emptyset$, $B \neq \emptyset$ so that $A \cup B = Z$ there is $x \in A$ and $\emptyset \neq B' \subseteq B$ so that $\{x\} \cup B' \in \mathfrak{C}$ or $y \in B$ and $\emptyset \neq A' \subseteq A$ so that $A' \cup \{y\} \in \mathfrak{C}$.

Theorem 2.19. *(S0), (S1), (S2), and (SX) is equivalent with (c0), (c1) and (cX).*

Proof. It will be convenient to express (SX) in its negated form: If $(A, B) \notin \mathfrak{S}$ then there is $x \in A$ so that $(\{x\}, B) \notin \mathfrak{S}$ or $y \in B$ so that $(A, \{y\}) \notin \mathfrak{S}$. We assume (S0), (S1), (S2), and (c0), (c1), respectively.

Now suppose (SX) holds and consider $Z \in \mathcal{C}$ and pair of subsets of Z so that $A \neq \emptyset$, $B \neq \emptyset$ so that $A \cup B = Z$. By definition, $(A, B) \notin \mathfrak{S}$ and hence, by (SX), there is $x \in A$ so that $(\{x\}, B) \notin \mathfrak{S}$ or $y \in B$ so that $(A, \{y\}) \notin \mathfrak{S}$. By Fact 2.14 there is $Z' \in \mathcal{C}$ with $B' := Z' \cap B \neq \emptyset$ and $x \in Z'$ or $Z'' \in \mathcal{C}$ with $A' := Z'' \cap B \neq \emptyset$, i.e., (cX) holds.

Conversely, suppose (cX) holds. Consider a pair of sets $C, D \in 2^X$ so that $(C, D) \notin \mathfrak{S}$. Then by Fact 2.14 there is a $Z \in \mathcal{C}$, $Z \subseteq C \cup D$ with $A := Z \cap C \neq \emptyset$ and $B := Z \cap D \neq \emptyset$. By (cX) there is $x \in A$ and $\emptyset \neq B' \subseteq B \subseteq D$ with $Z' := \{x\} \cup B' \in \mathcal{C}$ or $y \in B$ and $\emptyset \neq A' \subseteq A \subseteq C$ with $Z'' := \{y\} \cup A' \in \mathcal{C}$. If (SX) does not hold, then both $(\{x\}, D) \in \mathfrak{S}$ and $(C, \{y\}) \in \mathfrak{S}$, which by (S1) implies $(\{x\}, B') \in \mathfrak{S}$ and $(A', \{y\}) \in \mathfrak{S}$, a contradiction. □

Axiom (cX) implies that every connected set with more than 4 points contains a strictly smaller connected set comprising at least 2 points.

In [19] the following, weaker condition has been investigated:

(c3) If $Y, Z \in \mathcal{C}$ and $Y \cap Z \in \mathcal{C}$ then there is $z \in Y \cap Z$ so that $Y \cup \{z\} \in \mathcal{C}$ and $Z \cup \{z\} \in \mathcal{C}$.

Corollary 2.20. *Suppose (S0), (S1), (S2), i.e., (c0) and (c1) holds then (cX) – or equivalently (SX) – implies (c3).*

Proof. Consider $Z, Z', Z'' \in \mathcal{C}$ so that $Z' \cup Z'' = Z$. By (cX) there is w.l.o.g. $x \in Z'$ and $Y \in Z''$ so that $\{x\} \cup Y \in \mathcal{C}$. By (c1) this implies $\{x\} \cup Z'' \in \mathcal{C}$ and $\{x\} \cup Z' = Z' \in \mathcal{C}$ by definition. □

We note that (c3) does not imply (cX) because there is, in general no guarantee that any division of a connected set $Z \in \mathcal{C}$ into two non-empty sets $A \cup B = Z$ is such that A or B contains a connected set.

3. Integral Connectivity Spaces and Wallace Separations

3.1. Disjunctive Separations

In most settings it is natural to require that singletons are connected, i.e., to add the axiom

(c2) $\{x\} \in \mathcal{C}$ for all $x \in X$

to the definition of connectivity spaces. In terms of the connectivity openings, it is equivalently expressed as

(o5) $\{x\}[x] = \{x\}$.

We remark that (o5) can be expressed equivalently as "$x \in A[x]$ for all $x \in A$ and all $A \in 2^X$." Thus the sets $\{A[x] | x \in A\}$ form a partition of A.

It is natural therefore, to complete a basis $\mathcal{B}$ by adding all singletons to $\mu'(\mathcal{B})$, i.e., $\mu(\mathcal{B}) := \mu'(\mathcal{B}) \cup \{\{x\} | x \in X\}$. It is obvious that $\mu(\mathcal{B})$ is the smallest collection of subsets of X satisfying (c0), (c1), and (c2).

Consider a relation $\mathfrak{S}$ on $2^X \times 2^X$ that satisfies (S0), (S1), (S2), as well as

(S3) $(X, Y) \in \mathfrak{S}$ implies $X \cap Y = \emptyset$.

Such symmetric, hereditary, disjunctive relations on $2^X \times 2^X$ were considered already in [12] and are known as *Wallace separations* [14, 15]. The results given above have been shown in the disjunctive setting already in this classical literature. We call $(X, \mathfrak{S})$ a *Wallace separation space* if $\mathfrak{S}$ satisfies (S0), (S1), (S2), and (S3).

Lemma 3.1. *Suppose $\mathcal{C}$ satisfies (c0) and (c1) and $\mathfrak{S}$ is the corresponding (maximal) symmetric separation. Then $\mathfrak{S}$ is a Wallace separation, i.e., $\mathfrak{S}$ satisfies (S3), if and only if $\mathcal{C}$ satisfies (c2).*

Proof. Suppose $\mathfrak{S}$ is disjunctive (S3). Then, by Lemma 2.9 $\{x\}$ is $\mathfrak{S}$-connected for all $x \in X$ since for all $(A, B) \in \mathfrak{S}$ with $x \in A \cup B$ and $A \cap B = \emptyset$, either $x \notin A$ or $x \notin B$. Conversely, suppose $\{x\} \in \mathcal{C}$ for all $x \in X$. Then $(A, B) \in \mathfrak{S}$ if $x \in A \cup B$ implies $x \notin A$ or $x \notin B$. Thus $A \cap B = \emptyset$. □

A connectivity space satisfying (c2) is called *integral*. Maximal Wallace separation spaces and integral connecitivity spaces are equivalent.

Lemma 3.2. [40, 13] *If $\mathfrak{S}$ satisfies (S2), and (SX), i.e., the corresponding closure function w is pointwise symmetric (R0), then $\mathfrak{S}$ is disjunctive (S3) and only if (X, w) is enlarging (K2).*

The maximal separations satisfying (S0), (S1), (S2), (S3) and (SX) thus correspond exactly to the *neighborhood spaces*, i.e., the closure spaces satisfying (K0), (K1), and (K2).

Fact 3.3. [42, Thm. 5.2] *An isotone closure space (X, w) is connected if and only if it is not the disjoint union of two closed (or two open) sets.*

3.2. Connectedness of Generalized Topologies sensu Császár

A "generalized topological space" (GTS) in the sense of Császár [6] consists of a set system $\mathcal{O} \subseteq 2^X$ so that (i) $\emptyset \in \mathcal{O}$ and (ii) $O_\iota \in \mathcal{O}$ for $\iota \in I$ implies $\bigcup_{\iota \in I} O_i \subseteq \mathcal{O}$. The elements of $\mathcal{O}$ are regarded as generalized open sets, their complements are closed sets. For each $A \subseteq X$ denote by $j(A)$ the union of all open sets contained in A and by $k(A)$ the intersection of all closed sets containing A. By construction, $k : 2^X \to 2^X$ is a generalized closure operator satisfying (K0), (K1), (K2), and (K4). The associated interior operator is $j(A)$ satisfies $j(A) = X \setminus k(X \setminus A)$.

A set $A \subseteq X$ is
semi-open, $A \in \mathcal{O}^\sigma$, iff $A \subseteq kj(A)$;
pre-open, $A \in \mathcal{O}^\pi$, iff $A \subseteq jk(A)$;
α-open, $A \in \mathcal{O}^\alpha$, iff $A \subseteq jkj(A)$;
β-open, iff $A \in \mathcal{O}^\beta$, $A \subseteq kjk(A)$ [43].

Since $kjkj = kj$ and $jkjk = jk$ [6] we see that jk, kj, jkj, kjk are idempotent (K4). A GTS is said to be *χ-connected* if there are no two non-empty subsets $U, V \in \mathcal{O}^\chi$ so that $U \cup V = X$, see e.g. [7, 8, 9] for a systematic analysis. This definition generalizes Fact 3.3. Properties of the Hausdorff-Lennes separation w.r.t. χ are studied e.g. in [44].

In the most general setting, [7] considers an isotonic function $\gamma : 2^X \to 2^X$. Then $\mathcal{O}^\gamma = \{A|A \subseteq \gamma(A)\}$ is a GTS. Call a set γ-closed if $X \setminus A \in \mathcal{O}^\gamma$ and let k_γ be the corresponding generalized closure operator, which again satisfies (K0), (K1), (K2), and (K4). Two sets U and V are called γ-separated if they are Hausdorff-Lennes separated w.r.t. k_γ, i.e., $k_\gamma(U) \cap V = U \cap k_\gamma(V) = \emptyset$. Clearly, γ-separatedness defines a Wallace separation with corresponding Wallace function k_γ. The γ-connected set of [7] are exactly the connected sets w.r.t. to this separation. This includes, in particular, also the specific GTS $\mathcal{O}^\chi$ mentioned in the previous paragraph.

We note, finally, that [7] also shows directly that the γ-connected sets satisfy (c1) and that γ-connected components are well-defined. Furthermore, the γ-connected components of a γ-closed set $Q = k_\gamma(Q)$ are also γ-closed.

4. The Additivity Axiom

A key role in the theory of separations and proximities is played by the axiom of additivity.

(S4) $(A, B \cup C) \in \mathfrak{S}$ whenever $(A, B) \in \mathfrak{S}$ and $(A, C) \in \mathfrak{S}$

It was introduced already by Wallace [12], who called separation spaces $(X, \mathfrak{S})$ that satisfy (S0) through (S4) "s-spaces".

A key consequence of (S4) is the classical "decomposition theorem" for connected topological spaces [12, 45]

Theorem 4.1. *Let $(X, \mathfrak{C})$ be a connectivity space with a separation $\mathfrak{S}$ that satisfies (S4).*

(D) *$C, Z \in \mathfrak{C}$, $C \subseteq Z$, $Z \setminus C := M \cup N$ and $(M, N) \in \mathfrak{S}$ implies $C \cup M \in \mathfrak{C}$ and $C \cup N \in \mathfrak{C}$.*

Proof. We follow [12, Thm.4.5.ii]. Suppose $C \cup M \notin \mathfrak{C}$, i.e., there is $(A, B) \in \mathfrak{S}$ with $A \neq \emptyset$, $B \neq \emptyset$, and $A \cup B = C \cup M$. Since $C \in \mathfrak{C}$, Lemma 2.9 implies at least one of $A \cap C$ and $B \cap C$ is empty. W.l.o.g., we assume $A \cap C = \emptyset$, thus $A \subseteq M$, and $(A, N) \in \mathfrak{S}$. From this and $(A, B) \in \mathfrak{S}$ we conclude using axiom (S4) that $(A, B \cup N)\mathfrak{S}$. But $Z = C \cup M \cup N = A \cup B \cup N$ is connected, thus either A or $B \cup N$, and hence B, must be empty. Thus $C \cup M \in \mathfrak{C}$. Connectedness of $C \cup N$ is shown analogously. □

A variant of the Decomposition Theorem (D) can be stated in terms of connected sets only. It serves as an axiom in [19]:

(c4) Suppose $C, Z \in \mathfrak{C}$, $C \subseteq Z$ and suppose W is a connected component of $Z \setminus C$. Then $Z \setminus W \in \mathfrak{C}$.

Lemma 4.2. *Suppose $\mathfrak{C}$ is an integral connectivity space. Then property (D) is equivalent to (c4).*

Proof. Set $U = (Z \setminus C) \setminus W$. By construction $Z \setminus C = U \cup W$ and Thm. 2.16 implies $(U, W) \in \mathfrak{S}$. Furthermore we have $Z = C \cup W \cup U$. Condition (D) thus implies that $U \cup C = Z \setminus V \in \mathcal{C}$ and $V \cup C = Z \setminus U \in \mathcal{C}$. Repeating the argument we see that $Z \setminus \bigcup_{\iota\in J} W_\iota \in \mathcal{C}$ for any $J \subset I$, where W_ι, $\iota \in I$, are the connected componentes of $Z \setminus C$. Thus $C \cup W_\iota$ is connected. Since a bipartition $M \dot{\cup} N = Z \setminus C$ into $\mathfrak{S}$-separated sets M and N does not $\mathfrak{S}$-separate connected components, we may write $M = \bigcup_{\iota\in J} W_\iota$ and $N = \bigcup_{\iota\in I\setminus J} W_\iota$. By (c1) and connectedness of $C \cup W_\iota$ both N and M are connected. □

A variant of (c4) is used additivity axiom in [19]:

(c4') Let $A, B, Z_i \in \mathcal{C}$ for all $i \in I$ and suppose $\bigcup_{i\in I} Z_i \in \mathcal{C}$. Then there is $J \subseteq I$ so that $A \cup \bigcup_{j\in J} Z_j \in \mathcal{C}$ and $\bigcup_{j\in I\setminus J} Z_j \cup B \in \mathcal{C}$.

Note that we may assume, as in [19], that the sets A, B, and Z_i, $i \in I$ are pairwise disjoint since overlapping sets can just as well be unified if this is not the case initially.

Lemma 4.3. *Suppose $(X, \mathcal{C})$ satisfies (c0), (c1) and (c2). Then (c4') is equivalent to condition (c4).*

Proof. Suppose (c4') holds and let $Z_i \in \mathcal{C}$, $i \in I$ be the connected components of $Y \setminus Z$ except W. By (c2) we have $W \cup \bigcup_{i\in I} Z_i = Y \setminus Z$. By (c4') there is $J \subseteq I$ so that $Z \bigcup_{j\in J} Z_j \in \mathcal{C}$ and $\bigcup_{j\in I\setminus J} Z_j \cup W \in \mathcal{C}$. Since W is a connected component of $Y \setminus Z$ no larger subset of $Y \setminus Z$ is connected, thus $J = I$, i.e., $Z \cup \bigcup_{j\in I} Z_i = Z \cup (Y \setminus Z) \setminus W = Z \setminus W \in \mathcal{C}$.

To see the converse implication, let $A, B, Z_i \in \mathcal{C}$, $i \in I$ and assume $Y := A \cup \bigcup_{i\in I} Z_i \cup B \in \mathcal{C}$ and let W be a connected component of $Y \setminus A$ containing B, which is necessarily of the form $W = B \cup \bigcup_{j\in J}$ for some $J \subseteq I$. The statement of the lemma then implies that $Y \setminus W = A \cup \bigcup_{j\in I\setminus J} \in \mathcal{C}$, i.e., (c4') holds. □

Note that in the absence of (c2) we have only the implications (c4) $\implies$ (c4').

A weaker form of the additivity axioms (c4) and (c4') is

(c4") If $Z_i \in \mathcal{C}$ for $i \in I = \{1, ..., n\}$ and $\bigcup_{i=1}^{n} Z_i \in \mathcal{C}$, then for every nonempty $J \subset I$ there is $j \in J$ and $k \in I \setminus J$ so that $Z_j \cup Z_k \in \mathcal{C}$.

Since n is finite we can view $V := \{Z_i | 1 \le i \le n\}$ as the vertices of a graph $G := G(\{Z_i\})$ with edges $i \sim k$ whenever $Z_i \cup Z_k \in \mathcal{C}$. Condition (c4) means that there is an edge across every vertex cut. Thus (c4) is equivalent to graph-theoretic connectedness of G. In particular, for every pair of sets Z_i, Z_j there is a path $P = (i =: i_0, i_1, \ldots, i_\ell := j)$ in G such that $Z_h \cup Z_{h+1} \in \mathcal{C}$ and any connected subgraph of G with vertex set V' corresponds to a collection $\{Z_i | i \in V'\}$ with

$\bigcup_{i\in V'} Z_i \in \mathcal{C}$. In the following we will occasionally make use of this graph-theoretic interpretation of (c4).

Fact 4.4. *Axiom (c4") is equivalent to the following statement*

(c4"') *If* $Z_i \in \mathcal{C}$ *for* $i = 1,\dots,n$ *and* $\bigcup_{i=1}^{n} Z_i \in \mathcal{C}$*, then there is a permutation* π *with arbitrary choice of* $\pi(1)$ *so that* $\bigcup_{j=1}^{k} Z_{\pi(j)} \in \mathcal{C}$.

Proof. Given $G(\{Z_i\})$, the order π can be constructed by a breadth-first-search on $G(\{Z_i\})$ starting with $\pi(1)$, which exists as an immediate consequence of the graph-theoretic connectedness of $G(\{Z_i\})$.

To see the converse, we proceed by induction. For $n = 1, 2$ property (c4") is trival. In the general step we know that $G(\{Z_i\}_{i<n})$ is connected. Choose $\pi(1) = n$. Then by assumption $Z_n \cup Z_{\pi(2)} \in \mathcal{C}$, i.e, $\{Z_n, Z_{\pi(2)}\}$ is an edge in $G(\{Z_i\})$, and hence $G(\{Z_i\})$ is connected. □

Both properties (c") and (c"') have been described e.g. in [45].

Lemma 4.5. *(c4') implies (c4").*

Proof. Let $Z_i \in \mathcal{C}$ for $i = 1, ..., n$ and $Z := \bigcup_{i=1}^{n} Z_i \in \mathcal{C}$. We proceed by induction. For $n = 2$, (c4") is satisfied trivially. For $n = 3$, suppose $Z_1 \cup Z_2 \in \mathcal{C}$. Now set $A = Z_1$ and $B = Z_2$. Then (c4') implies that $Z_1 \cup Z_3 \in \mathcal{C}$ or $Z_2 \cup Z_3 \in \mathcal{C}$, the graph G with three vertices in the proof of fact 4.4 is connected . Now suppose $n > 3$. By (c4') there is a subdivision of Z into two connected sets both comprising fewer constituents Z_i. Subdividing each of these further we obtain at least one connected set comprising exactly two constituents, i.e., there is $j \neq k$ so that $Z' := Z_j \cup Z_k \in \mathcal{C}$. Thus we can write Z as a union of $n-1$ connected sets, and hence by assumption the corresponding graph G' on $n-1$ vertices is connected. the vertex representing Z' is connected to a non-empty set $L = \{l_1, \dots l_u\}$ of vertices representing constituents Z_l. Thus $Z_j \cup Z_k \cup Z_l \in \mathcal{C}$ for all $l \in L$ and hence $Z_l \cup Z_j \in \mathcal{C}$ or $Z_l \cup Z_k \in \mathcal{C}$. Thus there is a path between l' and l'' for all $l', l'' \in L$, and hence the n-vertex graph G is connected if and only if the $n-1$-vertex graph G' is connected. As in the proof of fact 4.4, connectedness of G implies that (c4") is satisfied. □

Note that in the absence of (c2) we have only the implications (c4) $\Longrightarrow$ (c4') $\Longrightarrow$ (c4").

For $n = 3$ we may rephrase (c4") in the following form.

Fact 4.6. *Suppose (c0), (c1), and (c4) holds,* $Z', Z'', Z'' \in \mathcal{C}$ *and* $Z' \cup Z'' \cup Z'' \in \mathcal{C}$*. Then at least two of the unions* $Z' \cup Z''$*,* $Z' \cup Z'''$*, and* $Z'' \cup Z'''$ *are connected.*

Note that Fact 4.6 does not imply (c4").

Theorem 4.7. *Let* $(X, \mathcal{C})$ *be an integral connectivity space, i.e., (c0), (c1), and (c2) hold. Then (c4') is equivalent to (S4) for the corresponding separation space.*

Proof. We already have seen that (S4) implies (D), (c4), and (c4') and that (c4), (c4'), and (D) are equivalent in integral connectivity spaces. Therefore, suppose that (D) but not (S4) is satisfied.

(S4) can be rephrased as $(A, B) \notin \mathfrak{S}$ implies $(A, U) \notin \mathfrak{S}$ or $(A, V) \notin \mathfrak{S}$ for all U, V so that $U \cup V = B$. Hence if (S4) fails we can find a subdivision of B so that $(A, U) \in \mathfrak{S}$ and $(A, V) \in \mathfrak{S}$. By Fact 2.14, $(A, B) \notin \mathfrak{S}$ implies there is $Z \subseteq A \cup B$ with $Z \in \mathcal{C}$, $A' = Z \cap A \neq \emptyset$ and $(U \cup V) \cap Z \neq \emptyset$. By (c2) we can decompose A', U, and V into connected components A_ι, U_κ, V_λ. By (c4") we can stepwise remove connected components of A', U, and V from $Z = A' \cup U \cup V$ so that the left-over set remains connected. In this process we either obtain a connected subset that comprises (i) only connected component(s) of A and either U or V, or that is of the form $Z' = A_1 \cup U_1 \cup V_1$, where A_1, U_1 and V_1 are connected components. In the first case we conclude immediately that $(A', U') \notin \mathfrak{S}$ or $(A', V') \notin \mathfrak{S}$, where U' and V' are connected components of U and V, respectively. In the alternative case (ii) we recall that (c4) implies (c4"), whence we may use Fact 4.6 to conclude that at least two of the three unions $A_1 \cup U_1$, $A_1 \cup V_1$ and $U_1 \cup V_1$ are connected so that $(A_1, U_1) \notin \mathfrak{S}$ or $(A_1, V_1) \notin \mathfrak{S}$. Heredity of $\mathfrak{S}$ now implies $(A, U) \notin \mathfrak{S}$ or $(A, V) \notin \mathfrak{S}$, i.e., property (S4) cannot fail. □

Theorem 4.8. *If* $(X, \mathcal{C})$ *is an additive, integral connectivity space, then (c3) implies (cX).*

Proof. Let $Z \in \mathcal{C}$ and $A \cup B = Z$. As in the proof above we can argue that Z contains a connected set Z' so that $A' = Z' \cap A$ and $B' = Z' \cap B$ are $\mathcal{C}$-connected. By (c3) there is a $z \in A' \cup B' \in Z' \subseteq Z$ so that $A' \cup \{z\} \in \mathcal{C}$ and $B' \cup \{z\} \in \mathcal{C}$. Either $A' \cup \{z\} \neq A'$ or $B' \cup \{z\} \neq B'$, i.e., the assertion of (cX) holds. □

It follows that the "connectologies" defined in [19] are equivalent to pretopological spaces.

We remark that the equivalence of (c4), (c4'), and (c4") is claimed in [19] alluding to Kuratowski's book [45], without hinting at a proof, however. We suspect that properties (c4") and (c4"') are in general strictly weaker than (c4) and (c4').

We close this section by linking (S4) with the corresponding property of the Wallace closure.

Theorem 4.9. [40] *If* $\mathfrak{S}$ *satisfies (S2), and (SX), i.e., the corresponding closure function* w *is pointwise symmetric (R0), then* $\mathfrak{S}$ *is additive (S4) iff* (X, w) *is sub-additive (K3).*

Proof. Let $x \in A$ and $B, C \subseteq X$ so that $(\{x\}, B \cup C) \notin \mathfrak{S}$. By (S4) $(\{x\}, B) \notin \mathfrak{S}$ or $(\{x\}, C) \notin \mathfrak{S}$. By definition of w, thus $w(B \cup C) \subseteq w(B) \cup w(C)$, i.e., w satisfies (K3). □

5. Some Additional Properties of Interest

5.1. Idempotency

Several axioms that are of interest in the context of proximity spaces, see e.g. [41]. These might also be of interest in the more general setting of Wallace separation spaces and (integral) connectivity spaces.

Of particular interest are axioms that render the corresponding Wallace function idempotent. In this context, two properties are of particular interest:

(S5) For all $A, B \subseteq X$, $(\{x\}, A) \notin \mathfrak{S}$ whenever $(\{x\}, B) \notin \mathfrak{S}$ and $(\{y\}, A) \notin \mathfrak{S}$ for each $y \in B$.

(C) $(x, A) \in \mathfrak{S}$ implies $(x, w(A)) \in \mathfrak{S}$

Lemma 5.1. [40, Thm.4] *Suppose $\mathfrak{S}$ is Wallace separation satisfying (SX). The w is idempotent (K4) if and only if $\mathfrak{S}$ satisfies (S5).*

Axiom (C) was introduced as axiom S.VI. already in [12]. It can replace (S5) as a condition for idempotency of w if $\mathfrak{S}$ is an additive Wallace separation, i.e., if (S0), (S1), (S2), (S3), and (S4) holds [46].

5.2. Separation Axioms

Separation axioms can be naturally phrased in terms of $\mathfrak{S}$. Since we pre-suppose symmetry, i.e., (R0), the weakest meaningful separation axiom is

(T1) $(\{x\}, \{y\}) \in \mathfrak{S}$ whenever $x \neq y$. Equivalently. $\{x, y\} \notin \mathfrak{C}$ if $x \neq y$

(T2) For every two distinct points $x, y \in X$ there are sets $U, V \in 2^X$ so that $U \cup V = X$, $(\{x\}, U) \in \mathfrak{S}$, and $(\{y\}, V) \in \mathfrak{S}$.

Fact 5.2. *Let $(X, \mathfrak{S})$ be a Wallace separation space. Then (T2) implies (T1).*

Proof. Suppose (T2) holds. Then $(\{x\}, U) \in \mathfrak{S}$ and (S3) implies $x \notin U$ and hence $x \in V$. From $(\{y\}, V) \in \mathfrak{S}$ and heredity (S1) we conclude $(\{y\}, \{x\}) \in \mathfrak{S}$. Symmetry (S2) now implies (T1). □

Fact 5.3. *Let $(X, \mathfrak{S})$ be a Wallace separation space satisfying (T1) and (T2), respectively. Then the corresponding closure space (X, w) satisfies the well-known Fréchet (T_1) and Hausdorff (T_2) separation axioms, respectively.*

Proof. Axiom (T1) implies that $y \notin w(\{x\})$ for $y \neq x$, i.e., $w(\{x\}) = \{x\}$ for all $x \in X$. This is one of the many equivalent versions of the Fréchet separation axiom.

Now suppose (T2) holds. Set $N := X \setminus U$ and $M := X \setminus V$. Since $U \cup V = X$ we have $N \cap M = \emptyset$. Furthermore, $(\{x\}, X \setminus M) \in \mathfrak{S}$ is equivalent to $x \notin w(X \setminus M)$, i.e., $x \in X \setminus w(X \setminus M) = i_w(M)$, the interior of M, hence M is a neighborhood of x. One shows analogously that N is a neighborhood of y, i.e., any pair of disjoint points x and y has disjoint neighborhoods in (X, w). This is the usual phrasing for the Hausdorff separation axiom. □

5.3. Efremovič's Axiom

Proximity is the complement of separation, i.e., two sets $A, B \in 2^X$ are "near" iff $(A, B) \notin \mathfrak{S}$. A proximity space in the sense of Efremovič [47] is equivalent to an additive Wallace separation that in addition satisfies the separation property (T1) and the axiom

(S6) $(A, B) \in \mathfrak{S}$ implies that there is $U \subseteq X$ such that $(A, U) \in \mathfrak{S}$ and $(X \setminus U, B) \in \mathfrak{S}$.

Instead of Efremovič's axiom (S6) the following condition of normality is often used in the literature:

(S6') $(A, B) \in \mathfrak{S}$ implies that there are sets $U, V \in 2^X$ so that $U \cup V = X$, $(A, U) \in \mathfrak{S}$ and $(B, V) \in \mathfrak{S}$.

Fact 5.4. *If $(X, \mathfrak{S})$ is a Wallace separation space then (S6) and (S6') is equivalent.*

Proof. (S6) is obtained from (S6') by setting $V := X \setminus U$, i.e., (S6) implies (S6'). Now suppose (S6') and pick $U' \subseteq U$ and $V' \subseteq V$ with $U' \cap V' = \emptyset$, i.e., $V' = X \setminus U'$. By heredity we have $(A, U') \in \mathfrak{S}$ and $(B, V') \in \mathfrak{S}$ as desired. □

In a proximity space, the Wallace function is idempotent (and hence defines a topology). Furthermore, it is well known that in a topological space, i.e., when $\mathfrak{S}$ satisfies (S0) through (S5) as well as (SX), then (S6) is equivalent to complete regularity [41]. In general, the Wallace function of a proximity space, which satisfies (S0) through (S4) and (S6) is completely regular.

6. Catenous Functions

Let $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a function between two connectivity spaces. Following [19] we say that f is *catenous* if $A \in \mathcal{C}_X$ implies $f(A) \in \mathcal{C}_Y$ for all $A \in 2^X$. It follows directly from the definition that the concatenation of catenous functions is again catenous.

Denote by $\nvDash$ the totally disconnected space on two points, i.e., the space $\{0, 1\}$ in which only $\emptyset$, $\{0\}$, and $\{1\}$ are connected. Note that we may regard $\nvDash$ also as neighborhood space since $w(\emptyset) = \emptyset$, $w(\{0\}) = \{0\}$, $w(\{1\}) = \{1\}$, and $w(\{1, 2\}) = \{1, 2\}$.

A classical theorem in point set topology asserts that X is connected if and only if every continuous function $X \to \nvDash$ is constant. In the realm of connectivity space we have the obvious analog:

Lemma 6.1. *An integral connectivity space $(X, \mathcal{C})$ is connected if and only if every catenous function $f : (X, \mathcal{C}) \to \nvDash$ is constant.*

Proof. If $A \subseteq X$ is a connected set and f is not constant, then $f(A) = \{0, 1\}$, i.e., not connected, contradicting that f is catenous. Thus f must be constant on every connected set. Any function $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ that is constant on connected components of X is obviously catenous as long as $(Y, \mathcal{C}_Y)$ is an integral connectivity space. □

Table 1. Summary of axioms. Properties of $(X, \mathfrak{S})$ and $(X, \mathfrak{C})$ are equivalent. For properties from (S4)/(c4) on, equivalence usually depends on (S0) to (S3), i.e., (c0) to (c2). If (SX)/(cX) holds in addition $(X, \mathfrak{C})$ and $(X, \mathfrak{S})$ are defined by a closure function.

Space	$\mathfrak{S}$	$\mathfrak{C}$	$w : 2^X \to 2^X$ (cX)/(SX)
connectivity space	(S0), (S1)		(K0), (K1)
symmetry	(S2)	(c0), (c1)	(R0)
disjunctivity	(S3)	(c2)	(K2)
addititivity	(S4)	(c4)	(K3)
idempotency	(S5)	–	(K4)
	(T1)	(T1)	Fréchet
	(T2)		Hausdorff
Efremovič	(S6)	–	completely regular

The restriction of a catenous function to a subspace remains catenous.

Interestingly, continous functions between topological spaces are catenous, but the converse is not true in general [19].

7. Discussion

In this contribution we have summarized a variety of independent approaches to axiomatizing connectedness in point set topology. We have focussed in particular on the close connection between separation spaces *sensu* Wallace and direct axiom systems for connected sets. Generalized closure space form an important special subclass, characterized by a single axiom (SX). Key properties of the closure (Wallace function) track the most important characteristics of separation functions, see Tab. 1. In particular, Wallace separations and integral connectivity spaces correspond to neighborhood spaces, while additive Wallace separations and additive connectivity spaces generalize pre-topologies. Proximity spaces form an even more specialized subclass.

The present contribution primarily aims at collecting and integrating the available basic results on generalized connectivity structures. Along the way, a variety of interesting research questions has appeared. For instance, it seems worth while to investigate generalizations of axioms of separations and regularity to connectivity spaces and their separation relations in such a way that the Wallace function w has prescribed separation or regularity properties [48, 49]. Furthermore, we have not discussed important constructions such as product spaces. In many details and combinations of axioms, finally, relationsships between connectivity spaces, separations relations, and Wallace functions remain to be elucidated.

Originally, this work was motivated by realization that connectedness rather than other topological constructions is the key ingredient for understanding fitness landscapes. In this context, as in the case of image processing, the special case of a finite set X is of most direct interest. We close the contribution therefore with a brief discussion of finite integral connectivity spaces, i.e., we assume that singletons are connected.

For finite X (c4) and (c4") are obviously equivalent, and hence every connected set Z is a union of finitely many disjoint singletons, whence we can break it down in connected pairs. In other words, a finite additive connectivity space is a finite graph Γ with vertex set X and edges defined as the two-element subsets of $\mathcal{C}$. As an immediate consequence property (cX) is satisfied automatically. The Wallace function of a singleton $\{x\}$ is thus simply its graph theoretic neighborhood, $w(\{x\}) = \{y \in X | \{x, y\} \in \mathcal{C}\}$. Additivity thus implies that $w(A) = \bigcup_{x \in A} w(\{x\})$. Since additive connectivity spaces are simply the finite undirected graphs there is little to be gained by starting with abstract connectedness.

As the example of recombination-based search spaces in [25] shows, however, much less obvious structures arise when additivity cannot be assumed, see also [34]. Such non-additive concepts of connectedness arise in particular in the context of combinatorial optimization when population-based heuristics implicitly define the structure of the search space, but also in the context of convexities, which in general also lack additivity.

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