The Ghost in the Machine: Basin of Attraction Fields of Disordered Cellular Automata Networks

Andrew Wuensche

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Basin of Attraction Fields of Disordered Cellular Automata Networks

Andrew Wuensche
Santa Fe Institute
wuensch@santafe.edu
contact address: 48 Esmond Road, London W4 1JQ, UK
tel 081 995 8893, fax 081 742 2178, email 100020.2727@compuserve.com
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Abstract

The basin of attraction fields of local 1-D cellular automata (CA) were presented in The Global Dynamics of Cellular Automata [1]. This paper extends the investigation to disordered CA networks (randomly wired/mixed rule), a very general class of discrete dynamical systems in which local CA form a special subset.

A general direct reverse algorithm is presented for generating the pre-images of any global state for disordered CA networks. Computation is many orders of magnitude faster than exhaustive testing. This allows the construction of transient trees or branches, basins of attraction, or the entire basin of attraction field, which represents the system's global dynamics and hierarchical categorisation of state space.

Contrasting the behaviour of local and disordered CA suggests that local wiring is a necessary condition for virtual automata to emerge in the CA's space-time pattern, the basis of CA models of artificial life. On the other hand, disordered CA networks, with their vast behaviour space, may serve as models of the activity of semi-autonomous groups of inter-connected neurons in the brain. Different wiring/rule schemes result in different field structure, suggesting a paradigm for emergent brain-like computation based on the categorisation of input; the basin of attraction field (the ghost in the machine) categorises input hierarchically at many levels, and may serve as a dynamical mind model for understanding cognitive processes such as memory and learning.

A learning algorithm is outlined which enables the network to learn any number of new pre-images to any given global state (and to forget old ones) by small adjustments to the network's wiring/rule scheme. This opens up the possibility of sculpting a basin of attraction field to achieve any desired structure. Disordered CA networks may thus offer a new approach to neural network models for brain-like computation.
1 Introduction

Basin of attraction fields allow the behaviour of cellular automata (CA) to be seen in the context of the global dynamics of the system; not only the unique trajectory of the system's future, but also the potentially many merging trajectories that could have constituted the system's past. A comprehensive survey of the basin of attraction fields of CA with local architecture, constructed by means of a direct reverse algorithm, was presented in *The Global Dynamics of Cellular Automata* [1]. In this paper the investigation is extended to disordered CA networks, where both the wiring scheme and the rule at each cell may differ.

Such systems have been studied by Stuart Kauffman (random Boolean networks [2-4]). Systems with various degrees of random wiring, but with a single rule (the same at all cells), have been studied by Crayton Walker (random nets, random structure CA [5-8]), and also by Wentian Li (non-local CA [9]). Disordered cellular automata are a very general class of discrete dynamical systems in which local CA form a special subset.

1.1 CA Dynamics

Cellular Automata are discrete space/time dynamical systems which evolve by the iteration of a specified procedure. The space of a CA is a potentially infinite D-dimensional array. A global state of the array is the pattern resulting from the individual values assigned to each array "cell" from a finite range of permitted values. In this paper the array size is taken as finite and the value range is generally taken to be binary. The total number of distinct global states is given by $G = k^L$, where $L$ is the array size and $k$ the value range. The set of $G$ global states constitute the CA's state space.

The dynamics of the system consists of the updating of the global state in discrete time steps, by the synchronous updating of the values of all cells. At each time step, each "target" cell updates its value as a function of the values of a set of "reference" cells at the previous time step, whose location may be specified by a wiring diagram.

The CA evolves along a trajectory consisting of a succession of global states, by the iteration of the global updating procedure (the transition function). Provided that the transition function is invariant and the system is autonomous (closed and noise free), then the system is deterministic. In fact the system may be regarded as semi-autonomous, in the sense that a global initial state must be imposed or perturbed from outside to set the system going along a new trajectory. It is also useful for the system to be capable of communicating its internal cell states at each time step to the outside.

In the case of 1-D CA, a trace through time may be made which completely de-
scribes the CA’s evolution from a given initial configuration. This is portrayed as rows of successive global states of the array, the space-time pattern (see figs. 7 and 8). Space-time patterns represent a deterministic sequence of global states evolving along one particular path within a basin of attraction, familiar from continuous dynamical systems. In a finite array, the path inevitably leads to a state cycle (the attractor cycle, or attractor). Other sequences of global states typically exist leading to the same attractor. The set of all possible paths leading to the same attractor, including the attractor itself, make up the basin of attraction. CA basins of attraction are thus composed of merging trajectories of global states linked according to their dynamical relationships, and will typically have a topology of branching trees rooted on attractor cycles.

Separate basins of attraction typically exist within state space. A CA transition function will, in a sense, crystallise state space into a set of basins of attraction, known as the basin of attraction field. The basin of attraction field is a mathematical object which constitutes the dynamical flow imposed on state space by the transition function. If represented as a graph the field is an explicit global portrait of a CA’s entire repertoire of behaviour. It includes all possible space-time patterns.

1.2 Depicting Basin of Attraction Fields

In this paper, basin of attraction fields are portrayed as computer generated diagrams (state transition graphs, networks of attraction [1]), in the same format as presented in The Global Dynamics of Cellular Automata [1]. Global states are represented by nodes, or by the state’s binary or decimal expression at the node position. Nodes are linked by directed arcs, which constitute transitions between global states.

Each node will have zero or more incoming arcs from nodes at the previous time-step (pre-images), but because the system is deterministic, exactly one outgoing arc (the “out degree”=1). Nodes with no pre-images have no incoming arcs, and are called garden of Eden states. The number of incoming arcs is referred to as the degree of pre-imaging [1] (or the “in degree”).

The make-up of a typical basin of attraction is illustrated by the state transition graph, fig.1 (it is part of the basin field shown in fig.2). In the graphic convention [1], the length of transition arcs decreases with distance away from the attractor, and the diameter of the graphic representation of the attractor asymptotically approaches an upper limit with increasing period, so that attractor cycles are drawn with approximately the same diameter irrespective of the number of nodes in the attractor. The
Figure 1: A basin of attraction (state transition graph) with 736 nodes. Evolution proceeds inwards from garden of Eden states to the attractor, then clockwise. Randomly wired, single rule, 3-neighbour rule 108, $L = 15$. The basin is indicated within the basin of attraction field in fig.2.

The forward direction of transitions is inward from garden of Eden states to the attractor, which is the only closed loop in the basin, and then clockwise around the attractor cycle.

Typically, the vast majority of nodes in a basin of attraction field, or a single basin of attraction, lie on transient trees outside the attractor cycle. A transient tree is the set of all paths from garden of Eden states leading to a particular state on the attractor cycle (an attractor state or node). A transient branch is the set of all paths from garden of Eden states leading to a state within a transient tree. A transient is one particular path from an arbitrary state in the transient tree leading to the attractor state. In all cases the attractor state itself is excluded from the definition.

Construction of a single basin of attraction poses the problem of finding the complete set of pre-images of every global state that is linked together in that basin. The trivial solution, exhaustive testing of the entire state space, becomes impractical in terms of computer time as the array size increases beyond modest values.

A reverse algorithm for local CA, that directly computes the pre-images of any global state, with an average computational performance many orders of magnitude
Figure 2: The basin of attraction field. The basin in fig.1. is indicated. Randomly wired, single rule, 3-neighbour rule 108, L=15. The field consists of 7 basins of attraction. The total number of states in each basin is as follows: 9100, 8136, 3788, 5520, 3220, 2268, 736. Wiring to the pseudo-local neighbourhood (see section 2) is shown on the right.

faster than exhaustive testing, was presented in [1]. In section 3, a general direct reverse algorithm for disordered CA is presented.

1.3 Overview

Section 2 looks at the hierarchy of possible CA architectures. At one extreme, local architecture is embedded in a regular space with periodic boundary conditions. Randomising local wiring (non-local CA) progressively renders the notion of space and dimension meaningless. Mixing rules (mixed-rule CA) breaks up spatial continuity. Disordered CA networks (non-local and mixed-rule) combine these properties. The behaviour space of disordered CA networks is shown to be vastly greater than for local CA.

Section 3 introduces a direct general reverse algorithm that computes the set of pre-images of a global state for disordered CA networks (with any wiring/rule scheme); the algorithm is many orders of magnitude faster than exhaustive testing. This allows the basin of attraction and the basin of attraction fields to be constructed.

Section 4 contrasts local and disordered CA. Local CA, (a regular spatial continuum
with the same "laws of physics" at all locations), is shown to be essential for CA models of artificial life, based on emergent virtual automata [13], periodic propagating and interacting structures within a CA space.

Such space dependent structures cannot emerge in disordered CA networks where space has been disrupted. However, disordered networks, because of their vast behaviour space, suggest a model of the activity of a semi-autonomous population of interconnected neurons in the brain. Different wiring/rule schemes result in different field structure suggesting that basin of attraction fields may serve as a paradigm for emergent brain-like computation. The basin of attraction field represents a hierarchical categorisation of state space, and therefore a categorisation of input, at many levels. Basin of attraction fields (the ghost in the machine) may thus serve as a dynamical mind model for understanding cognitive processes such as memory and learning.

Section 5 shows that the disordered CA network may be re-drawn as a brain-like model in 3-dimensions. A development of the brain-like model as a nested hierarchy of networks of networks, with an implicit web of interacting basin of attraction fields representing a mind model, is suggested.

In section 6, a learning algorithm is outlined which enables the network to learn any desired set of pre-images to a given global state by adapting its wiring/rule scheme. Thus new basin field structures may be learned, and old structures forgotten. It may be conjectured that given any desired structure of a transient branch, transient tree, basin of attraction, or basin of attraction field, a wiring/rule scheme can be evolved by degrees that will result in progressively closer approximations to that structure. The network may thus be capable of universal computation. This may have applications as a new approach to current neural and connectionist network models for brain-like computation.

Classical neural network models with weighted connections are shown to be essentially equivalent to randomly wired, single rule, CA networks, where multiple connections to one cell represent weights.

2 Local and Disordered CA Architecture

The overall architectural parameters of CA construction may vary widely within the constraints of a deterministic system. Starting with the most general, the parameters can be progressively narrowed as follows.

The most permissive CA architecture, termed disordered, allows a different random
wiring set-up and rule at each cell, for example Stuart Kauffman’s random Boolean networks [2-4]. In randomly wired CA architecture (non-local [9]), different random wiring is allowed at each cell, but a single rule applies, the same for all cells. Such a system may be regarded as a population of cells, where the dimensions of the array are irrelevant.

Ordered CA architecture imposes not only the same rule, but the same (possibly random) wiring template at each cell, and implies a space with given dimension and geometry, and periodic boundary conditions. In ordered architecture, rotation equivalent global states are embedded in equivalent behaviour, and the rotation symmetry of global states, (and bilateral symmetry for symmetric rules) is conserved in CA evolution [1].

Local architecture further restricts the standard wiring to a closed zone of cells centred on the target cell, the local neighbourhood, to which conventional CA rules are applied. In order to apply conventional CA rules to a disordered CA network, and define precisely the network’s particular wiring/rule scheme, the notion of a pseudo-local neighbourhood will be employed, illustrated in fig.6.

2.1 Intermediate Architectures

There are many possible intermediate CA architectures between disordered and local. Pseudo-local neighbourhoods may be wired up in such a way that some wires are local
and some randomised. For example, Crayton Walker's *networks of Boolean functions* [5-8] may be said to have a 3-neighbour pseudo-local neighbourhood with a local centre wire and the 2 outer wires randomly connected. Randomised wiring may be restricted in a number of ways. It may be confined to a local periodic zone; randomised wiring may rule out more than one wire sampling the same cell, or some wiring positions may be excluded, for instance a cell sampling itself.

Randomising the wiring of a local CA in stages will progressively transform a structured space-time pattern to a random pattern. Fig.7a shows a space-time pattern where a complex 5-neighbour local CA rule has its wiring randomised in stages by randomly re-wiring 2% of all wiring connections at each stage. A CA with local wiring may have a heterogeneous rule mix (fig 8a). The rule mix may be totally random, or restricted to a given choice of rules. A non-exhaustive diagram of possible CA architectures is shown in fig.3.

### 2.2 Local CA Architecture

In the special case of ordered or local CA architecture, the relative position of the reference cells to each target cell is constant (and can therefore be specified by a single
Disordered CA architecture is shown as a 1-D array for convenience only, with a pseudo-local neighbourhood.

For this condition to hold, the array must have periodic boundary conditions, i.e. a circle of cells for 1-D, or a toroidal surface for 2-D. In local architecture, the set of reference cells form a closed neighbourhood centred on the target cell. The architecture of a local, 1-D CA is represented in fig. 4, with time-steps separated, and arranged in sequence from the top down. The 1-D line of L cells is arranged in a circle to illustrate periodic boundary conditions. The value of each cell at time \( t_1 \) depends on the values of the cells in the local neighbourhood at \( t_0 \), whose location is specified by a standard template or wiring diagram. The updating procedure is iterated for successive time-steps.

The total number of permutations of values in a neighbourhood of size \( n \) is given by,

\[
P_{\text{total}} = k^n \quad \text{...where } k \text{ is the value range.}
\]

A rule table (look up table) with \( k^n \) entries will specify the output of all neighbourhood permutations. The total number of distinct rule tables (global rules), or the size of rule space is given by,

\[
R_{\text{total}} = k^{kn}
\]
This paper generally limits the value range to binary, where \( k = 2 \). By convention [10], the rule table is arranged in descending order of the values of binary neighbourhood strings, or their decimal equivalents. For example, the rule table for \( n = 3 \) rules (elementary rules [10]) will appear as follows,

<table>
<thead>
<tr>
<th>Rule table</th>
<th>( T_7 )</th>
<th>( T_6 )</th>
<th>( T_5 )</th>
<th>( T_4 )</th>
<th>( T_3 )</th>
<th>( T_2 )</th>
<th>( T_1 )</th>
<th>( T_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>decimal equivalent</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>neighbourhoods</td>
<td>111</td>
<td>110</td>
<td>101</td>
<td>100</td>
<td>011</td>
<td>010</td>
<td>001</td>
<td>000</td>
</tr>
<tr>
<td>outputs (0 or 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The \( n = 3 \) rule table with \( 2^3 = 8 \) entries thus specifies a total of \( 2^{2^3} = 256 \) distinct rules, numbered from 0 to 255 according to the decimal equivalent of the rule table regarded as a binary number. A \( n = 5 \) rule table has \( 2^5 = 32 \) entries and \( 2^{32} = 4,294,967,296 \) distinct rules. Equivalence relationships among rules reduces effectively different behaviour to slightly more than \( R_{total}/4 \) equivalence classes; for \( n = 3 \) rules there are 88 [1,6]. Some rules can also be expressed as simple algorithms, or as totalistic rules [1,11].

### 2.3 Disordered CA Architecture

The architecture of a disordered CA network is represented by the diagram in fig.6. with time steps separated. The cells are shown as a 1-D array for convenience only. The value of each cell at \( t_1 \) depends on the values of a randomly located set of \( n \) reference cells at \( t_0 \). In order to define precisely the network's particular wiring/rule scheme, the target cell is shown wired to the reference cells from a pseudo-local neighbourhood. This simulates a closed local neighbourhood (as in a CA with local architecture), but each cell in the pseudo-local neighbourhood draws its value according to its wiring connection. The wiring scheme for each target cell may be different. A conventional CA rule will apply to the pseudo-local neighbourhood as for local rules.

### 2.4 Size of Behaviour Space

The number of alternative wiring/rule schemes may be derived as follows.

**Wiring.** The number of alternative wiring schemes for one cell is given by,

\[
W_{one\text{cell}} = L^n \quad \text{where} \quad L \text{ is the array size, } n \text{ the number of wires}
\]
The number of wiring schemes for the whole array (assuming the number of wires per cell is the same) is given by,

\[ W_{\text{array}} = (L^n)^L \]

**Rules.** The total number of alternative rules for one cell is given by,

\[ R_{\text{one cell}} = k^{kn} \]

If different cells are also allowed different rules, the total number of permutations of rules for the whole array is given by,

\[ R_{\text{array}} = (k^{kn})^L \]

**Wiring/Rules.** The total number of alternative wiring/rule schemes for the array, the size of behaviour space, is given by,

\[ WR_{\text{array}} = W_{\text{array}} \times R_{\text{array}} = (L^n)^L \times (k^{kn})^L \]

thus in a binary CA with an array of 16 cells, with 3 wires per cell,

\[ WR_{\text{array}} = (16^3)^{16} \times (2^8)^{16} = 2^{192} \times 2^{128} = 2^{320} \]

in a binary CA with an array of 16 cells, with 5 wires per cell,

\[ WR_{\text{array}} = (16^5)^{16} \times (2^{32})^{16} = 2^{320} \times 2^{512} = 2^{832} \]

Even for small systems, the behaviour space is vast, and increases at a multiple exponential rate with increasing \( L, n \) and \( k \).
3 The General Reverse Algorithm for Disordered CA Networks

Consider a disordered binary CA network of size \( L \), with a pseudo-local neighbourhood, \( n = 3 \). For convenience, the system is represented as a 1-D array, \( a_1, a_2, \ldots, a_L \).

Each cell, \( a_x \), may have a value of 0 or 1, and has a pre-set wiring/rule scheme (possibly selected at random).

The wiring scheme is given by, \( a_x(w_1, w_2, w_3) \) where \( w_1 \) is a number between 1 and \( L \) signifying the position of the wire connection from the first branch of the pseudo-local neighbourhood, and so on, as illustrated in the example below, where \( w_1 = 7, \ w_2 = 6 \) and \( w_3 = 3 \).

\[
\begin{array}{cccccccc}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_L \\
\text{pre-image...} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{wiring scheme...} & w_1, w_2, w_3
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{pseudo-local neighbourhood...}
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{known state...} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccccccc}
a_1 & a_2 & \ldots & a_x & \ldots & a_L
\end{array}
\]

The rule scheme is given by, \( a_x(T_7, T_6, \ldots T_0) \), the \( n = 3 \) rule table for the rule assigned to cell \( a_x \).

To derive the pre-images of an arbitrary global state, consider an empty array, consisting of empty cells, as a potential pre-image. The cells are empty because their values are unknown, and unallocated as either 0 or 1. Empty cells are denoted by the wildcard symbol "*", known cells (with value 0 or 1) are denoted by the symbol "o".

Consider a known CA state, \( a_1, a_2, \ldots, a_L \), and an empty pre-image state, \( P_1, P_2, \ldots, P_L \).

\[
\begin{array}{cccccccc}
\text{empty pre-image...} & P_1 & P_2 & \ldots & \ldots & P_L
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{wiring scheme...}
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{pseudo-local neighbourhood...}
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{known state...} & 0 & 0 & 0 & 0 & a_1 & \ldots & \ldots & \ldots & a_L
\end{array}
\]
Starting with the first cell of the known state, $a_1$, the valid pseudo-local neighbourhood values, consistent with the value of $a_1$, are assigned to separate copies of the empty pre-image according to the wiring scheme $a_1(w_1, w_2, w_3)$. As there will be a mix of 0s and 1s in the rule table, only some of the 8 possible pseudo-local neighbourhoods will be valid. If, say, 3 are valid, 3 partial pre-images (with some cells known, and some empty) will be generated. For example, given the $n = 3$ rule 193 at $a_1$, with a rule table as follows,

\[
\begin{array}{cccccccc}
111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\
\end{array}
\]

Rule table

\[
\begin{array}{cccccccc}
T_7 & T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

If $a_1 = 1$, then only 3 outputs match $a_1$, $T_7$, $T_6$ and $T_0$, corresponding to the neighbourhoods 111, 110 and 000. These valid neighbourhoods are allocated to 3 empty arrays according to the wiring scheme, say $a_1(3, 7, 1)$. Each of the 3 arrays now have some of their cells allocated as 0s or 1s, and are termed partial pre-images, as illustrated below.

\[
\begin{array}{cccccccc}
P_1 & P_2 & \ldots & \ldots & P_L \\
0 & * & 0 & * & * & * & 0 & * & * \\
0 & * & 1 & * & * & 1 & * & * \\
1 & * & 1 & * & * & 1 & * & * \\
\end{array}
\]

partial pre-image 3...
partial pre-image 2...
partial pre-image 1...
wireing scheme...
pseudo-local neighbourhood...
known state...

The procedure continues with the next cell of the known state, $a_2$. Say that the value of $a_2$ has 5 (out of 8) valid pseudo-local neighbourhoods in its rule table, $a_2(T_7, T_6, \ldots T_0)$. The pseudo-local neighbourhoods are allocated to 5 copies of each of the three partial pre-images that were generated at $a_1$, according to the wiring scheme $a_2(w_1, w_2, w_3)$.

If the allocation of a value to a given cell conflicts with a value already assigned to that cell, then the partial pre-image is rejected. Otherwise, the partial pre-image is added to the partial pre-image stack. The allocation will be valid if it is made to an empty cell, or a known cell with an equal value. Valid allocation increases the size of the partial pre-image stack, conflicts can reduce the size of the stack.
This procedure is repeated in turn for the remaining cells, \( \ldots a_3, a_4,\ldots a_L \) (in fact it may be carried out in any order). At each successive cell, more partial pre-images may be added to the stack, but also rejected. The size of the stack will typically vary as depicted in the diagram below. If the stack size is reduced to zero at any stage then the known state has no pre-images; it must be a *garden of Eden state.*

The final stack at \( a_L \) may still have empty cells, signifying that these cells are not sampled by any wiring connection. Final stack arrays with empty cells are replicated as many times as necessary so that all possible configurations at empty cell positions are represented. The resulting pre-image stack is the complete set of pre-images of the given known state, without duplication. This information is used to construct a transien branch, transient tree, basin of attraction, or the complete basin of attraction field, following the method described in [1].

An equivalent, but extended, procedure is used for \( n = 5 \) rules. In this case the wiring/rule scheme is specified for each cell in the array \( a_1, a_2,\ldots a_L \) as follows,

- the wiring scheme... \( a_x(w_1, w_2, w_3, w_4, w_5) \)
- the rule scheme..... \( a_x(T_{32}, T_{31}, T_{30}, \ldots T_0) \)

The general reverse algorithm works for any degree of disordered CA architecture. This of course includes local CA architecture (local wiring/single rule) which is a special subset of disordered CA. In principle, the algorithm will work for any size of pseudo-local neighbourhood, \( n \), and any value range, \( k \). The dimensions of a CA array are of no consequence. Provided that \( n \) is smaller than \( L \), the general reverse algorithm for generating pre-images is many orders of magnitude faster than the exhaustive testing of state space.
4 Local and Disordered CA Behaviour

A broad distinction can be made between the behaviour of local CA and disordered CA networks. Local CA support the emergence of structure in the CA's space-time pattern, including virtual automata. This behaviour depends on spacial continuity. By contrast, disordered CA networks cannot support such emergent structure, but have a vastly greater behaviour space, with potentially "brain-like" properties. Intermediate behaviour is evident when the CA's wiring is partly local and partly random.

4.1 Local CA

Local CA capable of supporting complex emergent behaviour, have sometimes been viewed as artificial universes with their own local physics [13]. The analogy is then made with the real world, which supports the emergence of life. In the real world, as in the local CA universe, the laws of dynamical change are the same at all locations; a given event is embedded in a spatial continuum and is subjected only to local interactions, or influences that taper off rapidly with distance.

Local wiring seems to be a necessary condition for the emergence of coherent interacting structures in a CA's space-time pattern (sometimes called virtual automata or information structures) from random initial conditions, which form the basis of CA models of artificial life[13]. Local wiring is essential for the stability of virtual automata, and the same rule at all points in space is essential to enable virtual automata to propagate through space.

Such emergent behaviour in local CA has been characterised as computation emerging spontaneously at a phase transition in rule space [14]. In disordered CA, where space is not significant, computation may nevertheless emerge in the absence of propergating structure.

An investigation of rule-space in terms of the topology of the basin of attraction field was investigated in [1]. Below the phase transition, one global configuration (often a short repeating sequence) quickly becomes dominant and overwhelms the environment of the evolving CA. Transients and attractor cycles are short and the convergence of the dynamical flow in state space, the degree of pre-imaging, is high. Above the phase transition, many structures continually emerge, interact and disintegrate. None are able to predominate in the evolving CA environment; space-time patterns will appear chaotic. Transients and attractor cycles are typically very long (analogous to strange attractors), the degree of pre-imaging is very low.
Complex rules are located at the phase transition, sometimes called the edge of chaos [14], where there is a balance between transient/attractor length and the degree of pre-imaging. A small subset of interacting periodic structures, virtual automata, take over the CA environment. The emergence and persistence of virtual automata, evolving within long transients, has been proposed as having potential for information processing, and implications for understanding the origin of life [14]. Artificial life models have been based on emerging virtual automata in 2-D local CA, for instance [13].

The basin of attraction fields of local CA exhibit a corresponding sense of order. This arises because rotation equivalent global states are embedded in equivalent behaviour, and the internal symmetries of global states are conserved in CA evolution, resulting in an underlying hierarchy in their global dynamics[1]. In a transient, rotational symmetry cannot decrease with time, in an attractor cycle rotation symmetry must remain constant. The same principles apply in the case of bilateral symmetry for symmetric rules.

The highest degree of rotation symmetry corresponds to the two uniform states, all 0s and all 1s, which cannot be upstream of any other state in a basin of attraction except each other. None of these principles apply to disordered CA networks, where the local wiring/rule scheme is disrupted; the concept of equivalent states no longer holds and all states, including the uniform states, may occur anywhere in the basin of attraction. In non-local CA (with a single rule) the uniform states alone remain special, as described above.

4.2 Disordered CA networks

Random wiring, even in CA with a single rule, prevents the emergence of virtual automata. Evolution generates patterns that appear entirely random until the onset of periodic behaviour at the attractor. The only semblance of order is the pattern density (the density of 1s), which tends to settle and fluctuate at characteristic mean-field levels, corresponding to the mean λ parameter [13,14] of the rule mix. Disordered CA networks, however, have a vast behaviour space as described in section 2. The network’s ability to learn (see following sections) may allow the basin of attraction field itself to become an emergent system capable universal computation, by its ability to categorise input hierarchically at many levels.

To illustrate how virtual automata are degraded by randomising local wiring, Fig.7a shows the space-time pattern of a complex binary, local, 1-D, 5-neighbour CA, with
n=5 rule 3162662612 with local and non-local wiring, $L=150$

a) The space-time pattern of a complex rule with local wiring from a random initial state. After about 240 time steps, the wiring scheme has been totally randomised.

b) Shows the pattern density (density of 1s) in successive global states analogous to an EEG.

c) The space-time pattern of the same rule from another random initial state. At the times indicated, 2% of the wires are cumulatively randomised.
an array size of 150. Virtual automata emerge from a random initial condition. After approximately 240 time steps, the wiring is totally randomised (but not the rule), resulting in the loss of all coherent structure. Fig.7b shows the pattern density against time, analogous to an EEG (electro-encephalogram), a measure of the mean excitatory state of a local pool of neurons in the brain. Fig.7c shows the space-time pattern of the same complex local CA from another initial random state. At the intervals indicated, 2% of the wiring scheme is cumulatively randomised (15 wires out of 750). Coherent structure is progressively eroded. Eventually, the space-time pattern will look like the lower half of fig.7a.

In a mixed rule CA, but with local wiring, periodic structures confined within vertical bands will rapidly emerge within the CA’s space (frozen islands and isolated islands of variable elements [4]), depending on the degree of rule heterogeneity. In disordered CA networks (both randomly wired and with mixed rules), space-time patterns appear chaotic, but with some vertical (space dependent) features. The mean-field pattern density corresponds to the mean $\lambda$ parameter of all the rules.

Fig.8a shows the space-time pattern of a CA with heterogeneous rule mix, but with local wiring. Frozen island structures emerge from a random initial condition. After approximately 240 time steps, the wiring has been totally randomised (but not the rule scheme), resulting in the loss of all coherent structure, though some vertical features are evident. Fig.8b shows the pattern density. Fig.8c shows the space time pattern of the same mixed rule, locally wired CA from the same initial random state. At the intervals indicated, 4% of the wiring scheme is cumulatively randomised (30 wires out of 750). Frozen island structure is progressively eroded. Eventually, the space-time pattern will look like the lower half of fig.8a.

The notion of the CA’s “space” loses significance as local architecture is progressively randomised. Randomising the wiring disrupts the dimensionality of space. A one-dimensional array of size $L$ becomes a population of $L$ cells where the location of each cell is of no consequence. If in addition each cell computes a different rule, the integrity of “space” is further disrupted.

The basin of attraction fields of disordered CA lack the orderly constraints of local CA architecture. However, as shown in section 2, their behaviour space, and thus field structure, is vastly greater than for local CA, because of the many possible permutations of wiring/rule schemes. One may conjecture that a wiring/rule scheme for a population of cells exists that would result in a basin of attraction field that will categorise any set of inputs into appropriate outputs; in section 5 a learning algorithm to achieve this is outlined. Separate basins in the field, and each node onto which dynamical flow has
Fig. 8

\( n=5 \) mixed rule CA with local and random wiring, \( L=150 \)

a) The space-time pattern with local wiring from a random initial state. After about 240 time steps, the wiring scheme has been totally randomised.
b) Shows the pattern density (density of 1s) in successive global states analogous to an EEG.
c) The space-time pattern of the same rule from the same random initial state as a). At the times indicated, 4% of the wires are cumulatively randomised.
converged, are potential information storage and recognition systems.

In figs.9-10 the basin of attraction fields of two examples of disordered CA networks are shown. In fig.9 the field consists of a single basin of attraction containing all states. In fig.10a, the field consists of 13 basins; fig 10b shows the third basin at a larger scale; fig.10c shows a *transient branch* in the basin, showing the decimal equivalents of the binary CA states at each node.

In such a system, any input will immediately initiate a dynamical flow along a unique chain of states (the transient trajectory). Each successive transient state categorises potential input from states that belong to its transient branch. At each successive state the transient branch may expand, increasing the proportion of states categorised, thus forming a hierarchy of categorisation. The highest level of categorisation of input is the attractor.

A small subset of states on the attractor of one particular basin will categorise all states belonging to the basin. The basin of attraction field, consisting of one or more basins, thus categorises input at many levels, dependent entirely on the wiring/rule scheme of the CA network. New information may be stored (re-categorised) by adjusting the rule/wiring scheme, analogous to learning. Once the system has learnt, it will react to input automatically, directly homing in on an appropriate sequence of outputs, without the need to sequentially search memory addresses as in conventional computer architecture. This is the process that operates in auto-associative neural network models, and possibly in the brain.

Disordered CA networks may thus serve as models of the activity of semi-autonomous populations of inter-connected neurons in the brain. Basin of attraction fields implicit in such networks may suggest a paradigm for information processing in the brain; the merging dynamical flow within separate basins in a basin of attraction field may serve as a mind model for understanding cognitive processes such as memory and learning.

Fig.9

The basin of attraction field of a disordered CA network, $L=13$, consisting of one single basin, with $2^{13}$ nodes.
The basin of attraction field of a disordered CA network, $L=13$, consisting of 13 separate basins. The total number of states in each basin is as follows: 3072, 80, 432, 1872, 628, 812, 34, 62, 114, 62, 512, 84, 428. Wiring to the pseudo-local neighbourhood (see section 2), and the rule scheme is shown in the table below.

<table>
<thead>
<tr>
<th>cell</th>
<th>wiring</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>3, 12, 6</td>
<td>87 - 01010111</td>
</tr>
<tr>
<td>2.</td>
<td>7, 11, 4</td>
<td>4 - 00000100</td>
</tr>
<tr>
<td>3.</td>
<td>3, 3, 1</td>
<td>194 - 11000100</td>
</tr>
<tr>
<td>4.</td>
<td>11, 3, 9</td>
<td>52 - 00110100</td>
</tr>
<tr>
<td>5.</td>
<td>8, 7, 5</td>
<td>235 - 11101011</td>
</tr>
<tr>
<td>6.</td>
<td>1, 8, 1</td>
<td>101 - 01100101</td>
</tr>
<tr>
<td>7.</td>
<td>12, 4, 13</td>
<td>6 - 00000110</td>
</tr>
<tr>
<td>8.</td>
<td>8, 6, 8</td>
<td>101 - 01100101</td>
</tr>
<tr>
<td>9.</td>
<td>9, 2, 6</td>
<td>6 - 00000110</td>
</tr>
<tr>
<td>10.</td>
<td>5, 1, 1</td>
<td>95 - 01011111</td>
</tr>
<tr>
<td>11.</td>
<td>2, 7, 1</td>
<td>74 - 01001010</td>
</tr>
<tr>
<td>12.</td>
<td>7, 8, 4</td>
<td>215 - 11010111</td>
</tr>
<tr>
<td>13.</td>
<td>1, 4, 7</td>
<td>189 - 10111101</td>
</tr>
</tbody>
</table>

The third basin in the basin of attraction field above (fig 10a) drawn at a larger scale. The transient branch detailed below is indicated.

The transient branch indicated in the basin of attraction above (fig 10b), showing the decimal equivalents of the binary CA states at each node.
5 A Brain-Like model

Each cell in the diagram in fig.6 may be represented by an idealised neuron as in fig.11.

5 dendrites sample the axons of other cells
5-neighbour pseudo-local neighbourhood
cell "computes" output according to its rule
output is sampled at axon at the next
time-step by between 0 and 5L dendrites

Fig.11
A single cell of a disordered CA network represented as an idealised neuron.

Fig.12
A 5-neighbour disordered CA network of 27 cells represented as an autonomous population of idealised neurons.

The cells in randomly wired CA are arranged in an orderly array for convenience only, but their location is arbitrary. One presumes that the actual location of neurons in the brain has been optimised by evolution to achieve high density by minimising the average length of connections which occupy space and consume resources. An example of such a system is re-drawn as a brain-like model in fig.12; a semi-autonomous population of 27 idealised neurons distributed in 3 dimensions. Each neuron receives a post-synaptic excitatory (1) or inhibitory (0) signal from a proportion of neurons in the population via its 5 dendrites, and computes a response signal to its axon according to its particular rule applied to its pseudo-local neighbourhood. (In the real brain, the
Fig. 13  A single cell of a disordered CA network represented as an idealised neuron with an overriding presynaptic contact from outside the system, and a projection axon to other centres.

Fig. 14  A disordered CA network represented as a semi-autonomous population of idealised neurons. Each member of the population has input and output links to other centres as in Fig. 13.

cresponse is said to depend on the dendritic morphology, synaptic microcircuitry, and intrinsic membrane properties [16]). The updating of axonic response is synchronous, and the process is iterated in discrete time steps; it may be that a local cluster of neurons that are simultaneously activated tend to correlate their activity.

A given wiring/rule scheme will map to a particular basin of attraction field, and may act as a dynamic recognition system for information stored as hierarchical categorisations of state space at all basin field nodes other than garden of Eden nodes. Information storage is thus implicit in the system’s hard wiring and rule allocation; in the case of the brain it is implicit in the brain’s “wetware”. Recognition is automatic and direct. Amending the stored information is achieved by adjustments to the wiring/rule scheme, analogous to adjusting synaptic connections and microcircuitry in the brain, which is thought to occur during learning [16].

The system is semi-autonomous because it must be capable of receiving input from outside, resetting the initial global state. In addition, neurons must have a feed forward channel to communicate their current internal state outside the system. Thus each neuron may be regarded as having one (or more) overriding posts-synaptic inputs from outside the population, and a pre-synaptic output to one (or more) targets outside the population. An idealised neuron, and a semi-autonomous population of idealised neurons, are represented in Fig 13 and 14.

A development of such a brain-like model might be elaborated further by inter-
connecting many semi-autonomous disordered CA networks, so that the output of a particular network constituted the overriding input of another. Networks would activate each other asynchronously, and at a slower frequency than a network's internal clock, because of the greater length of transmission axons as compared with local dendrites. Networks may prevent other networks reaching their attractor cycles, thus avoiding repetitive, possibly pathological behaviour.

Some networks would link directly to the external environment either as sensors or effectors. Other networks would reprocess information internally with feed forward and feedback connections. A mechanism for amending the wiring/rule scheme by small degrees would be built into the system to permit learning.

Such a nested hierarchy of networks of networks will have implicit in its particular pattern of connections at any given instant, a vastly more complex but intangible web of interacting basin of attraction fields capable of categorising and re-categorising information. An idealised diagram of a network of networks is shown in fig. 14a. Each network in the diagram may be composed of a sub-network in a nested hierarchy.

Whether or not such a model might be biologically plausible, it may be useful in its own right as a transparent connectionist computational system, where learning is equivalent to re-configuring the system's basin of attraction field. This is discussed in the next section.

Fig. 14a
A system of interconnected semi-autonomous CA networks, with feedback from the environment. Each network may consist of a nested hierarchy of networks. Updating between networks is asynchronous.
6 The Learning Algorithm

A CA network can learn (and also forget), by small adjustments to its wiring or rule scheme, particular transitions between global states. This allows the possibility of finding the appropriate wiring/rule scheme between cells to produce any desired set of pre-images to a given global state. It may be conjectured that given any desired structure of a transient branch, transient tree, basin of attraction, or basin of attraction field, a wiring/rule scheme can be evolved by degrees that will result in progressively closer approximations to that structure. This may have applications as a new approach to current neural and connectionist network models for brain like computation.

The network can learn either by rewiring or by mutations to the rule scheme. Rewiring is a more fundamental adjustment, as will be shown below, analogous to changes in a neuron’s synaptic connections so that the set of neurons sampled is slightly altered. Mutating the rule scheme is potentially a finer adjustment, analogous to changes in the topology of the dendritic tree, the microcircuitry of synaptic placements and intrinsic membrane properties; a neuron’s output is thought to result from a non-linear “computation” dependent on these factors [16].

Another possibility is the role played by the neuron’s cytoskeleton of microtubules which is said to provide the cell’s “internal nervous system”, and also to perform other vital functions including locomotion, synapse modulation and dendritic spine formation [18]. There appears to be no shortage of biological mechanisms to permit each neuron
to express a rich and flexible repertoire of computation, which is modelled by the rule table.

Before learning starts, a wiring/rule scheme must already be in place. The closer such a wiring/rule scheme is to the behaviour to be learnt, the better. The wiring/rule scheme may be selected from an “atlas” of basin of attraction fields, such as presented in [1], or in the worst case assumed at random. Suppose that the network is to learn the arbitrary global states $P_1, P_2, P_3, \ldots$ etc as the pre-images of the arbitrary global state $A$, as represented in fig. 15; two alternative learning algorithms are available.

![Diagram](image)

**Fig. 16**

a) Learning the global state $A$ as a pre-image of itself, making a point attractor. 

b) Learning the global state $A$ as a distant pre-image of itself, making a cyclic attractor.

### 6.1 Learning by re-wiring

Consider the given state $P_1$, $(P_{11}, P_{12}, \ldots P_{1L})$, that the network is to learn as a pre-image of the given state $A$, $(a_1, a_2, \ldots a_L)$. When $P_1$ is evolved forward by one time-step according to its current $n$-neighbour wiring/rule scheme, it will produce state $B_1$, $(b_{11}, b_{12}, \ldots b_{1L})$. If the choice of states $P_1$ and $A$ was random, then $B_1$ will probably have mismatches with $A$ in about $L/2$ locations.
Where there is a mismatch between $A$ and $B_1$, for instance at cell 1, one wire (or the minimum number of wires) is re-wired from cell 1’s pseudo-local neighbourhood to a position that will cause $b_1$’s output to be flipped (changed from 0 to 1 or vice versa). There will be about $L/2$ alternative choices of valid re-wiring positions for each element in the pseudo-local neighbourhood, thus $V = n \times L/2$ alternative re-wiring options (the re-wiring position may initially be selected at random). $b_1$ is now equal to $a_1$. The procedure is repeated for the other mismatches. $P_1$ is now a pre-image of $A$. 

If the network is to learn another given state $P_2$ as a pre-image of the same given state $A$, the procedure is repeated for $P_2$. However, for each mismatch, say at cell 1, any re-wiring position that would negate the previously learned pre-image $P_1$ is avoided, otherwise learning $P_2$ would cause the network to forget $P_1$. Again there are $V = n \times L/2$ alternative re-wiring options for each mismatch. 1/2 of these options would be likely to alter the pseudo-local neighbourhood values for $b_1$, 1/2 of which would be likely to give outputs of $b_1$ different from $a_1$. Therefore $V/4$ options would be likely to be invalid and $3V/4$ valid. For each successive pre-image to be learnt, provided the pre-images are random with respect to each other, with a Hamming distance of about $L/2$, the valid re-wiring options will be reduced to $3/4$ of the previously valid options. If the pre-images are close to each other in terms of Hamming distance, then the reduction in valid options at each step will be less and the capacity to learn will be greater.

The capacity of the network to learn new pre-images of a given state by re-wiring will thus depend on a number of factors: the original wiring/rule scheme, the similarity of the new pre-images, and the size of $L$ and $n$. Additional cells added to the network will increase its learning capacity. If the network’s capacity becomes exhausted, it will have additional capacity to learn distant pre-images, further upstream in the transient tree as in fig 15. Note that if the network learns the given state itself as its own pre-
image, this will result in a point attractor as in fig.16a; if the state itself is learned as a distant pre-image, this will result in a cyclic attractor with a period equal to the distance as in fig 16b.

6.2 Learning by Mutating the Rule Scheme

Consider the given state \( P1, (P1_1, P1_2, \ldots, P1_L) \), that the network is to learn as a pre-image of the given state \( A, (a_1, a_2, \ldots, a_L) \). When \( P1 \) is evolved forward by one time-step according to its current \( n \)-neighbour wiring/rule scheme, it will produce state \( B1, (b1_1, b1_2, \ldots, b1_L) \). If the choice of states \( P1 \) and \( A \) was random, then \( B1 \) will probably differ from \( A \) in about \( L/2 \) locations.

\[
\begin{array}{ccccccc}
  P1_1 & P1_2 & \ldots & \ldots & P1_L \\
  P1_1 & P1_2 & \ldots & \ldots & P1_L \\
  \text{aspiring pre-image } P1 \ldots & \text{wiring scheme…} & \text{pseudo-local neighbourhood…} & \text{actual state } B1 \ldots & \text{given state } A \ldots \\
  \text{mutate the rule at cell 1 to flip } b1_1 \text{ to match } a_1 \\
  \text{where there is a mismatch between } A \text{ and } B1, \text{ for instance at cell 1, the rule at cell 1 is mutated by flipping the bit in its rule table corresponding to } b1_1 \text{’s pseudo-local neighbourhood. } b1_1 \text{ is now equal to } a_1. \text{ The procedure is repeated for the other mismatches. } P1 \text{ is now a pre-image of } A. \text{ At each of the mismatch positions (of which there will be about about } \frac{L}{2}, \text{ see above), the rule at that position is mutated by one bit. For } n \text{-neighbour wiring this represents a change of } \frac{1}{2^n} \text{ in the rule table, ie. for } 3\text{-neighbour wiring } \frac{1}{8}, \text{ for } 5\text{-neighbour wiring } \frac{1}{32}. \text{ For larger } n \text{ this becomes a relatively finer alteration to the transition function. It was shown in [1] that a 1 bit mutation in a local } n\text{-neighbour rule resulted generally in a small change in the basin of attraction field.} \\
  \text{The procedure is repeated for the next pre-image, } P2. \text{ If there is a mismatch between } b2_1 \text{ and } a_1 \text{ then } b2_1 \text{’s and } b1_1 \text{’s pseudo-local neighbourhoods must be different. Therefore flipping the rule table corresponding to } b2_1 \text{’s pseudo-local neighbourhood to correct a mismatch cannot affect } b1_1. \text{ The surprising consequence of this is that the network can learn more pre-images without any risk of forgetting previously learnt pre-}
\end{array}
\]
images. There is no limit, other than the size of state space itself \((2^L)\), to the number of pre-images to any given state that may be learnt by the network.

An extreme example is to assign only the two trivial rules with rule tables consisting of all 0s or all 1s to a disordered CA network. Allocating these rules to cells in the network will create a point attractor with cell values of 0 for all-0 rules, and 1 for all-1 rules. All other states in state space will be garden-of-Eden pre-images of the point attractor.

Learning distant pre-images by mutating the rule scheme may cause pre-images “downstream” to be forgotten with a small probability of \(1/2^n\). Transient trees may therefore be learnt by the network, as can point and cyclic attractors as illustrated in figs.15 and 16. Combining learning by re-wiring and learning by mutating the rule scheme may result in a powerful method for categorising input, recognition, and cumulative learning in disordered CA networks.

### 6.3 Disordered CA Networks and Neural Networks

Sparsely connected neural networks with weighted connections may be re-interpreted as randomly wired CA networks with a large pseudo-local neighbourhood and a homogenous rule scheme. An example of a typical two-layer neural network is given below.

```
cell value (-1 or +1)    o o o o o o o o input
weighted connections (0 - 1)
standard threshold function... o o o o o o o o output
```

Each cell in the output layer has weighted connections (a real number between 0 and 1) to cells in the input layer, and computes an output according to a standard threshold function. A given output can be learned by altering the weights (and positions) of connections using various techniques such as back propagation.

This system may be re-interpreted as a disordered CA network by discretizing the system’s weights as in the example below, and applying a standard majority rule at all cells. As we have seen, however, any other arbitrary rule scheme is possible.
The neural network's weighted connections are replaced by multiple discrete connections to the pseudo-local neighbourhood. Other connectionist models may equally be capable of being re-interpreted as disordered CA networks, and their ability to categorise input explained by the basin of attraction field.

7 Conclusion

This paper has extended the ability to depict basin of attraction fields to disordered CA networks, a very general class of discrete dynamical systems which allow arbitrary wiring and rule allocation. A general reverse algorithm for generating the pre-images of a global state for any wiring/rule scheme has been presented, by means of which the fields may be constructed.

A learning algorithm has been outlined that permits the network to learn any number of new pre-images to any given global state (and to forget old ones), by adjusting its wiring/rule scheme. This opens up the possibility of sculpting basin of attraction fields to achieve any desired structure. The basin of attraction field is a vehicle for categorising information hierarchically at many levels. The possibility therefore exists of applying disordered CA networks to computational tasks, as a new approach to neural and connectionist network models.

A broad distinction has been made between local CA, and "brain-like" disordered CA networks. Coherent space-time structures, virtual automata which constitute CA models of artificial life, only emerge in local CA; disordered CA networks have a vastly greater behaviour space which may support emergent basin of attraction fields. A combination of disordered CA networks in a nested hierarchy of networks of networks have been proposed as a brain-like model. Such a model will have implicit in its connections a hidden and more complex intangible reality, a web of interacting basin of attraction fields representing a dynamical model of mind - the ghost in the machine.
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References


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