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# The Critical Line in Random Threshold Networks with Inhomogeneous Thresholds

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We calculate analytically the critical connectivity  $K_c$  of Random Threshold Networks (RTN) for homogeneous and inhomogeneous thresholds, and confirm the results by numerical simulations. We find a super-linear increase of  $K_c$  with the (average) absolute threshold  $|h|$ , which approaches  $K_c(|h|) \sim |h|^\alpha$  with  $\alpha \approx 2$  for large  $|h|$ , and show that this asymptotic scaling is universal for RTN with Poissonian distributed connectivity and threshold distributions with a variance that grows slower than  $|h|^\alpha$ . Interestingly, we find that inhomogeneous distribution of thresholds leads to increased propagation of perturbations for sparsely connected networks, while for densely connected networks damage is reduced. Further, damage propagation in RTN with in-degree distributions that exhibit a scale-free tail  $k_{in}^\gamma$  is studied; we find that a decrease of  $\gamma$  can lead to a transition from supercritical (chaotic) to subcritical (ordered) dynamics. Last, local correlations between node thresholds and in-degree are introduced. Here, numerical simulations show that even weak (anti-)correlations can lead to a transition from ordered to chaotic dynamics, and vice versa. Interestingly, in this case the annealed approximation fails to predict the dynamical behavior for sparse connectivities  $\bar{K}$ , suggesting that even weak topological correlations can strongly limit its applicability for finite  $N$ .

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## I. INTRODUCTION

Many systems in nature, technology and society can be described as complex networks with some flow of matter, energy or information between the entities the system is composed of; examples are neural networks, gene regulatory networks, food webs, power grids and friendship networks. Often, in particular when the networks considered are very large, many details of the topological structure as well as of the dynamical interactions between units are unknown, hence, statistical methods have to be applied to gain insight into the global properties of these systems. In this spirit, Kauffman [1, 2] introduced the notion of Random Boolean Networks (RBN), originally as a simplified model of gene regulatory networks (GRN). In a RBN of size  $N$ , each node  $i$  receives inputs from  $0 \leq k \leq N$  other nodes (with  $k$  usually either considered to be constant, or distributed according to a Poissonian with average  $\bar{K} \ll N$ ), and updates its state according to a Boolean function  $f_i$  of its inputs; the subscript  $i$  indicates that Boolean functions vary from site to site, usually assigned at random to each node. It was shown that RBN exhibit a percolation transition from ordered to chaotic dynamics at a critical connectivity  $\bar{K} = K_c = 2$ . Since interactions in RBN are asymmetric and hence a Hamiltonian does not exist, mean-field techniques have to be applied for analytical calculation of critical points, for example the so-called *annealed approximation* (AA) introduced by Derrida and Pomeau [3–5]. In the AA, random perturbations are applied to initial dynamical states, and random ensemble techniques are applied to determine whether the so-induced “damage” spreads over the network or not. Recent research has revealed many surprising details of RBN dynamics at criticality, e.g. super-polynomial scaling of the number

of different dynamical attractors (fixed points or periodic cycles) with  $N$  [6] (while Kauffman assumed it to scale  $\sim \sqrt{N}$  [1]), as well as analytically derived scaling laws for mean attractor periods [7] and for the number of frozen and relevant nodes in RBN [8, 9]. Similarly, it was shown recently that dynamics in finite RBN exhibits considerable deviations from the AA (that is exact only in the limit  $N \rightarrow \infty$ ) [10, 11]. Boolean network models have been applied successfully to model the dynamics of real biological systems, e.g. the segment polarity network of *Drosophila* [12], dynamics and robustness of the yeast cell cycle network [13], damage spreading in knockout experiments [14] as well as establishment of position information [15] and cell differentiation [16] in development. Other models explicitly evolve RBN topology according to local rewiring rules coupled to local order parameters of network dynamics (e.g., the local rate of state changes), and investigate the resulting self-organized critical state [17–19].

A drawback of RBN is the fact that, in spite of their discrete nature (which makes them easy to simulate on the computer in principle), the time needed to compute their dynamics in many instances scales exponential in  $N$  and  $\bar{K}$ , and often large statistical ensembles are needed for unbiased statistics due to the strongly non-ergodic character [20] of RBN dynamics. For this reason, there exists considerable interest in simplified models of RBN dynamics, as, for example, *Random Threshold Networks* (RTN), that constitute a subset of RBN.

In RTN, states of network nodes are updated according to a weighted sum of their inputs plus a threshold  $h$ , while interaction weights take (often discrete and binary) positive or negative values assigned at random. The critical connectivity, calculated by means of the AA, was found to deviate slightly from RBN [21–23]; this analysis was extended to RTN dynamics including stochastic update

errors [24]. In particular, it was found that phase transitions in RTN with scale-free topologies [24, 25] substantially differ from both RTN with homogeneous or Poissonian distributed connectivity and scale-free RBN [26]. Further, dynamics in finite RTN with  $k = \text{const.} = 2$  inputs per node recently was found to be surprisingly ordered, including, e.g., globally synchronized oscillations [27]. Other approaches, that apply learning algorithms as well as ensemble techniques, present evidence that information processing of static [28] or time-variant [29] external inputs is optimized at criticality in both RBN and RTN.

In this paper, we extend the theoretical analysis of RTN in a number of respects. First, we calculate the critical connectivity  $K_c$  for arbitrary thresholds  $h \leq 0$ , and generalize this derivation for the first time to inhomogeneously distributed thresholds  $h_i$  that can vary from node to node. This generalization, that introduces an additional level of complexity to RTN dynamics, is motivated by recent observations of strong variations in regulatory dynamics from gene to gene in real GRN, caused by, for example, the frequent occurrence of canalizing functions [20] and the abundance of regulatory RNA in multicellular organisms which strongly influence the expression levels and -patterns of (regulatory) proteins [30]. Using the AA and additional approximation techniques, we derive a general scaling relationship between critical connectivity  $K_c$  and (average) absolute node threshold  $|h|$ , and show that  $K_c(|h|)$  asymptotically approaches a unique power law  $K_c(|h|) \sim |h|^\alpha$  with  $\alpha \approx 2$  for large  $|h|$ . Evidence is presented that this asymptotic scaling law is universal for RTN with Poissonian distributed connectivity and threshold distributions with a variance that grows slower than  $|h|^\alpha$ .

Further, we establish that damage propagation functions of RTN with homogeneous thresholds  $|h|$  and of RTN with inhomogeneous thresholds with the same *average*  $|\bar{h}| = |h|$  intersect at characteristic connectivities  $K_d(|h|) > K_c(|h|)$ , which implies that for  $\bar{K} < K_d$ , random distribution of thresholds tends to increase damage, while for  $\bar{K} > K_d$ , the opposite holds. Evidence is presented that  $K_d(|h|)$  converges to the same asymptotic scaling law as  $K_c(|h|)$ . Next, we analyze the dynamics of RTN with in-degree distributions that exhibit scale-free tails, while for small  $k$ , the distribution deviates from a power law (as frequently found in nature). Both analytically and numerically we establish that flat power-law tails lead to a transition from chaotic to ordered dynamics through damage suppression at highly connected in-degree "hubs". Last, we investigate the effect of correlations between thresholds  $h_i$  and in-degree  $k_i$ , while keeping all other network parameters constant. We find that even small positive correlations can induce a transition from supercritical (chaotic) to subcritical (ordered) dynamics, while anti-correlations have the opposite effect.

## II. RANDOM THRESHOLD NETWORKS

A Random Threshold Network (RTN) consists of  $N$  randomly interconnected binary sites (spins) with states  $\sigma_i = \pm 1$ . For each site  $i$ , its state at time  $t+1$  is a function of the inputs it receives from other spins at time  $t$ :

$$\sigma_i(t+1) = \text{sgn}(f_i(t)) \quad (1)$$

with

$$f_i(t) = \sum_{j=1}^N c_{ij} \sigma_j(t) + h_i, \quad (2)$$

where  $c_{ij}$  are the interaction weights. If  $i$  does not receive signals from  $j$ , one has  $c_{ij} = 0$ , otherwise, interaction weights take discrete values  $c_{ij} = \pm 1$ ,  $+1$  or  $-1$  with equal probability. In the following discussion we assume that the threshold parameter takes integer values  $h_i \leq 0$  [32]. Further, we define  $\text{sgn}(0) = -1$ . [33] The  $N$  network sites are updated *synchronously*. Notice that we depart from the well-studied case  $h_i = \text{const.} = 0$  in two respects:  $h_i$  can take arbitrary values  $h_i \leq 0$ , and it can differ from node to node (inhomogeneous thresholds).

Let us now have a closer look on network topology. Let  $\bar{K}$  be the *average connectivity*, i.e. the average number of inputs (outputs) per site, and let us assume that each interaction weight has equal probability  $p = \bar{K}/N$  to take a non-zero value. Further, let us consider the limit of sparsely connected networks with  $\bar{K} \ll N$ . Under these assumptions, the statistical distribution  $\rho_k$  of in- and out-degrees follows a Poissonian:

$$\rho_k = \frac{\bar{K}^k}{k!} e^{-\bar{K}}. \quad (3)$$

Further, we study the case where in- and out-degree distributions differ: while the out-degree is still distributed according to a Poissonian, the in-degree distribution exhibits a power-law tail, i.e.

$$\rho_{k_{in}} \propto k^{-\gamma} \quad (4)$$

with  $2 \leq \gamma \leq 4$ .

## III. CALCULATING THE CRITICAL LINE

### A. Uniform threshold $h < 0$

We start with the simplest case and assume that all network sites have identical integer threshold values  $h_i \equiv h \leq 0$ . The case  $h > 0$  is not studied here, as it may lead to the pathological outcome of nodes set to an active state  $\sigma_i = +1$ , though they receive only inhibitory inputs  $c_{ij} < 0$ .

Let us first calculate the probability for damage spreading  $p_s(k)$ , i.e. the probability that a node with

$k$  inputs changes its state, if one of its input states is flipped. A straight-forward extension of the combinatorial analysis carried out in [23] for the special case  $h = 0$  yields

$$p_s(k, |h|) = k^{-1} \cdot 2^{-(k-1)} \cdot \left[ (k + |h| + 1) \cdot \left( \frac{k}{\frac{k+|h|+1}{2}} \right) + (k - |h| + 1) \cdot \left( \frac{k}{\frac{k-|h|+1}{2}} \right) \right] \quad (5)$$

$$= 2^{-(k-1)} \left( \frac{(k-1)}{\frac{k+|h|+1}{2}} \right) \quad (6)$$

for odd  $k - |h|$  with  $k > |h|$ , and

$$p_s(k, |h|) = k^{-1} \cdot 2^{-(k-1)} \cdot \left[ (k - |h|) \cdot \left( \frac{k}{\frac{k-|h|}{2}} \right) + (k + |h| + 2) \cdot \left( \frac{k}{\frac{k+|h|+2}{2}} \right) \right] \quad (7)$$

$$= 2^{-(k-1)} \left( \frac{(k-1)}{\frac{k+|h|}{2}} \right) \quad (8)$$

for even  $k - |h|$  with  $k > |h|$  (for a detailed derivation, please refer to appendix A). Notice that Eqs. (6) and (8) are similar, yet not identical to the corresponding relations derived in [24] for RTN with probabilistic time evolution; in particular, for the RTN with *deterministic* dynamics as studied here, the relation  $p_s^{\text{odd}}(k) = p_s(k-1)$  holds only for the special case  $|h| = 0$ , whereas for  $|h| > 0$ ,  $p_s(k)$  exhibits an oscillatory behavior (Fig. 1).

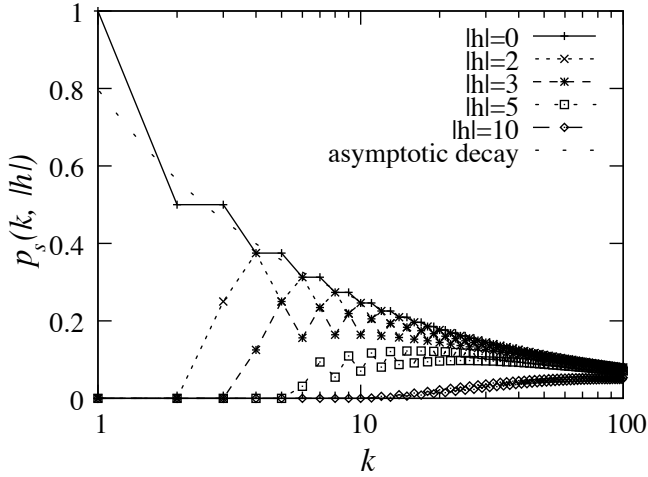


FIG. 1: Probability  $p_s(k, |h|)$  of damage propagation, for different values of the threshold  $|h|$ , as a function of the number of inputs  $k$ . For large  $k$ , the curves asymptotically approach  $p_s \sim 1/\sqrt{k}$  (dashed line). Notice the oscillatory behavior for  $|h| > 0$ .

If we know the statistical distribution function  $\rho_k$  of the in-degree, the average damage spreading probability

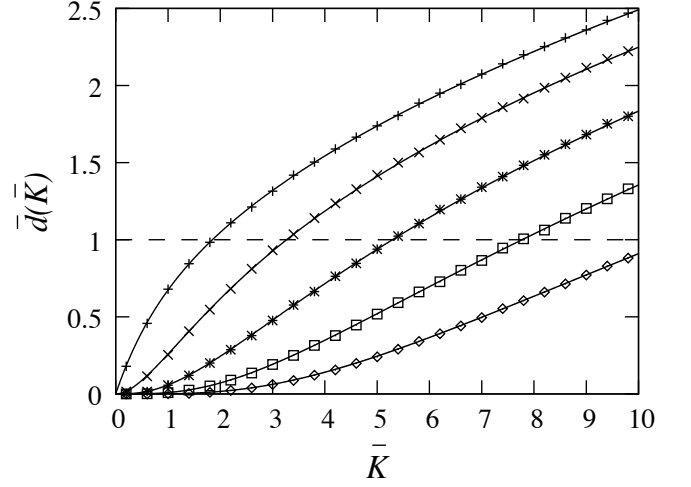


FIG. 2: Expectation value  $\bar{d}$  of damage one time step after a one-bit perturbation, as a function of the average connectivity  $\bar{K}$ , and different (homogeneous) thresholds  $|h|$  ( $|h| = 0$  (+),  $|h| = 1$  (x),  $|h| = 2$  (\*),  $|h| = 3$  ( $\square$ ),  $|h| = 4$  ( $\diamond$ )). Solid curves are the corresponding analytical results obtained from the annealed approximation.

then simply follows as [23]

$$\langle p_s \rangle = \sum_{k=|h|}^N \rho_k p_s(k+1, |h|), \quad (9)$$

where  $\langle \cdot \rangle$  indicates the average over the ensemble of all possible network topologies that can be generated according to the degree-distribution  $\rho_k$ . In the case of a Poisson distributed connectivity with average degree  $\bar{K}$ , it follows

$$\langle p_s \rangle(\bar{K}, |h|) = e^{-\bar{K}} \sum_{k=|h|}^N \frac{\bar{K}^k}{k!} p_s(k+1, |h|). \quad (10)$$

Let us now apply the so-called *annealed approximation* [3], which averages the effect of perturbations over the whole ensemble of possible network topologies and all possible state configurations; in this approximation, the expected damage  $\bar{d}$  after one update time step, given a one-bit perturbation at time  $t-1$  then follows as

$$\bar{d}(t+1) = \langle p_s \rangle(\bar{K}, |h|) \cdot \bar{K}, \quad (11)$$

where  $\bar{\cdot}$  denotes the average over all possible network topologies and all possible state configurations. If we apply a sufficiently large (but finite) upper limit  $N$  to the sum in Eq. (10), we can numerically evaluate this formula with any desired accuracy. Figure 2 shows the results for the first five values of negative  $h$  of RTN with Poissonian distributed connectivity, compared to measurements obtained from numerical simulations of large ensembles of randomly generated instances of RTN, indicating an excellent match between theory and simulation.

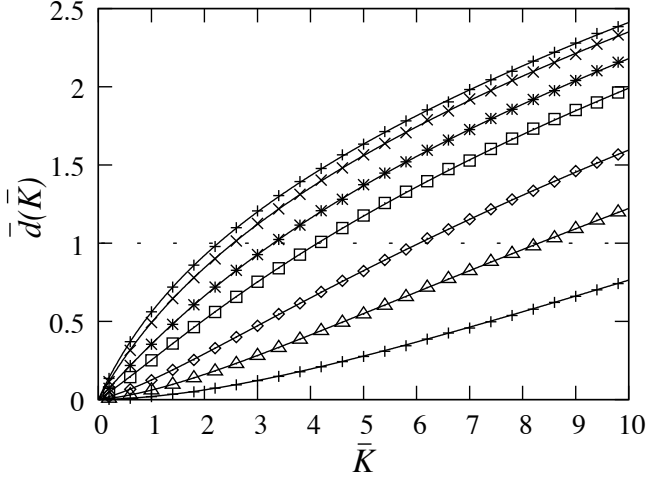


FIG. 3: Average damage  $\bar{d}(\bar{K})$  one time step after a one-bit perturbation, for Poisson-distributed connectivity with average degree  $\bar{K}$ , and Poisson-distributed negative thresholds with average absolute value  $|\bar{h}|$ ; points are data from numerical simulations of RTN (ensemble averages over 100000 different network realizations for each data point), lined curves are analytical solutions (annealed approximation). Numerical data were sampled for  $|\bar{h}| = 0$  (+),  $|\bar{h}| = 0.3$  (X),  $|\bar{h}| = 1.0$  (\*),  $|\bar{h}| = 1.5$  (squares),  $|\bar{h}| = 2.5$  (diamonds),  $|\bar{h}| = 3.5$  (triangles) and  $|\bar{h}| = 5.0$  (+).

### B. Poisson distributed thresholds

Let us now consider the more general case of non-uniform thresholds, i.e., networks where each site  $i$  has assigned an individual threshold  $h_i \leq 0$ . In the simplest case, we can imagine that the final thresholds resulted from iterated, random decrementsations (starting from  $h = 0$  for all sites), until a certain average threshold  $\bar{h}$  is reached - this process results in Poisson distributed thresholds  $h_i$ . If threshold assignment is independent from the (also Poisson distributed) in-degree, the probabilities for  $k$  and  $h$  simply multiply, and the resulting average damage propagation probability is

$$\langle p_s \rangle(\bar{K}, |\bar{h}|) = e^{-(\bar{K} + |\bar{h}|)} \sum_{|h|=0}^N \sum_{k=|h|}^N \frac{\bar{K}^k |\bar{h}|^{|h|}}{k! |h|!} p_s(k+1, |h|), \quad (12)$$

where  $|\bar{h}|$  is the average absolute threshold.

Figure 3 demonstrates that the expected damage  $\bar{d}_{t+1}(\bar{K}, |\bar{h}|)$  resulting from a one-bit perturbation at time  $t$ , as predicted from this annealed approximation over both degree- and threshold distribution, exhibits excellent agreement with the results obtained from numerical simulations of randomly generated RTN ensembles. It is an interesting question how the dynamics of RTN with inhomogeneous thresholds compares to RTN with homogeneous thresholds. Figure 4 shows  $\bar{d}(\bar{K})$  for RTN with different homogeneous  $|h| = \text{const.}$  and the corresponding inhomogeneous RTN with Poisson-distributed

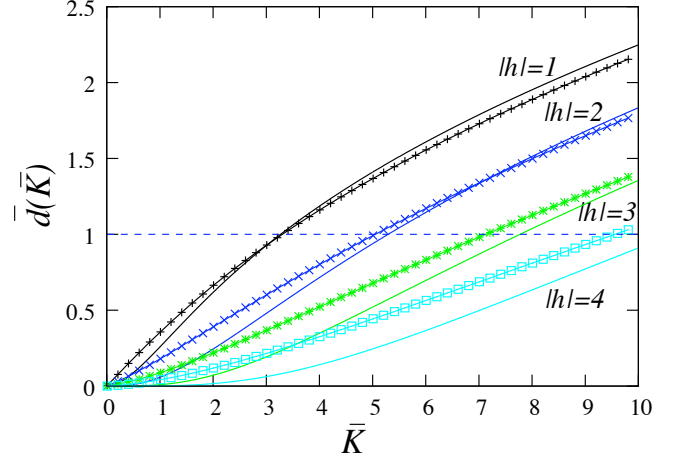


FIG. 4: Comparison of damage spreading in networks with homogeneous thresholds  $|h| = \text{const.}$  (solid lines, threshold values  $|h|$  as indicated) vs. networks with inhomogeneous thresholds distributed according to a Poissonian with the same average threshold  $|\bar{h}|$  (curves with data points,  $|\bar{h}| = 1$  (+),  $|\bar{h}| = 2$  (x),  $|\bar{h}| = 3$  (\*) and  $|\bar{h}| = 4$  (□); results obtained from the annealed approximation).

thresholds with the same average  $|\bar{h}| = |h|$ , as obtained from the annealed approximation. One observes that for small  $\bar{K}$ , the curves for RTN with inhomogeneously distributed thresholds are systematically above those of the corresponding homogeneous RTN, i.e., the randomization of node thresholds increases dynamical disorder - also, the critical connectivities  $K_c(|h|)$  (intersections with the line  $\bar{d} = 1$ ) are shifted to smaller values. However, one also realizes that the curves intersect in the supercritical phase at characteristic connectivities  $K_d(|h|)$ , i.e., for  $\bar{K} > K_d(|h|)$ , inhomogeneity in thresholds actually reduces damage.

### C. Universal scaling of the critical line

If we again assume a one-bit perturbation at time  $t$ , the critical line  $K_c(|h|)$ , that separates the ordered and the chaotic phase of RTN dynamics, is given by the condition

$$\bar{d}(t+1) = \langle p_s \rangle(K_c(|h|), |h|) \cdot K_c(|h|) = 1. \quad (13)$$

Again, we can apply Eq. (10) to solve this equation for arbitrary  $h \leq 0$ , however, numerical evaluation is almost impossible for  $|h| > 80$  due to exponentially diverging computing time caused by evaluation of the sum in Eq. (13) for large  $\bar{K}$  [34]. For estimation of the scaling behavior of  $K_c(|h|)$  for large  $|h|$ , we are interested in a good approximation that does not require summation over the whole network topology, and hence neglect the variation in  $k$ , considering damage propagation in the *mean field limit*  $k = \text{const.} \approx \bar{K}$  (for details, see Appendix B). Using the Stirling approximation

$$n! \approx n^n e^{-n} \sqrt{2\pi n}, \quad (14)$$

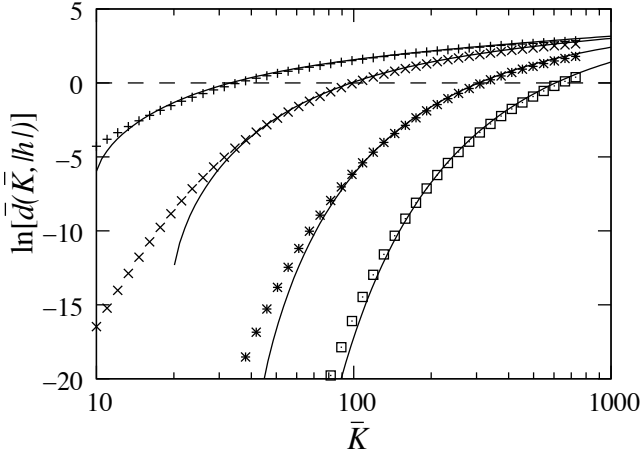


FIG. 5: Logarithm of the average damage,  $\ln[\bar{d}(\bar{K})]$ , as calculated from the annealed approximation, for different values of  $|h|$  ( $|h| = 10$  (+),  $|h| = 20$  (X),  $|h| = 40$  (\*) and  $|h| = 60$  ( $\square$ )). The corresponding solid curves are obtained from Eq. (15). For not too small  $\bar{K}$ , one finds that Eq. (15) approximates the true damage function very well.

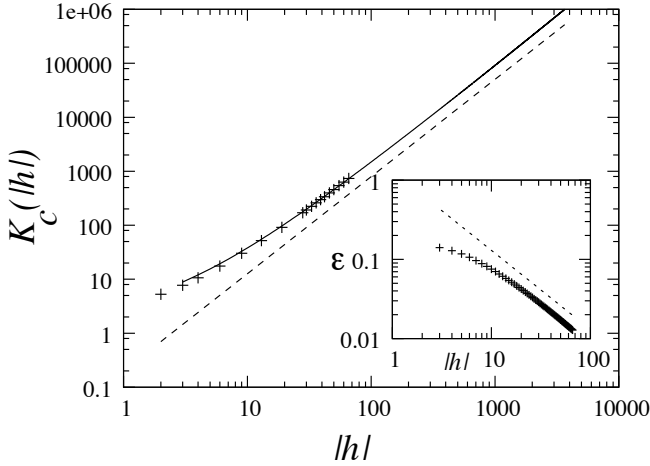


FIG. 6: Scaling behavior of the critical connectivity  $K_c(|h|)$  as a function of the (homogeneous) node threshold  $|h|$ , log-log-plot. Data points + are solutions obtained from the annealed approximation of Eq. (13), the solid curve is obtained from setting Eq. (15) to zero. A straight line (power law)  $\sim |h|^\alpha$  with  $\alpha = 1.8$  is shown for comparison. *Inset*: relative error  $\epsilon$  between the approximation of Eq. (15) and the result obtained from the annealed approximation, as a function of  $|h|$  (log-log-plot). For  $|h| \geq 15$ ,  $\epsilon$  vanishes  $\propto |h|^{-1}$ ; dashed line with slope  $-1$  shown for comparison.

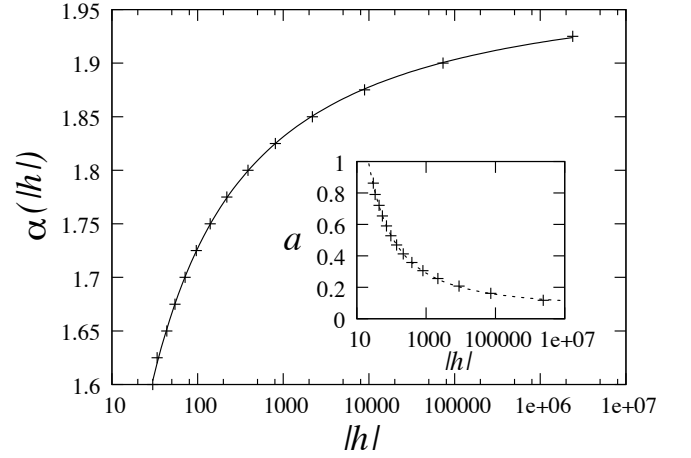


FIG. 7: Optimal exponents  $\alpha$  of power-laws  $K_c \approx a|h|^\alpha$  that approximate the scaling function  $K_c(|h|)$ , as shown in Fig. 6, as a function of  $|h|$ . For large  $|h|$ ,  $\alpha$  converges to a unique value  $\alpha_{\text{inf}} = 1.978 \pm 0.006$ . *Inset*: Power-law constant  $a$  as a function of  $|h|$ .

this leads to the following approximation for the logarithm of the damage:

$$\ln[\bar{d}(\bar{K}, |h|)] \approx \frac{1}{2} \left\{ \ln \bar{K} - \bar{K} \cdot \ln \left[ 1 - \left( \frac{|h|}{\bar{K}} \right)^2 \right] - |h| \ln \left[ \frac{\bar{K} + |h|}{\bar{K} - |h|} \right] \right\} + C \quad (15)$$

with  $C = \ln(\sqrt{2/\pi})$ ; solving this equation for

$$\ln[\bar{d}(K_c(|h|), |h|)] = 0 \quad (16)$$

then yields the critical connectivity  $K_c(|h|)$ . Figure 5 shows that this approximation is very accurate even for considerably small, finite  $|h|$ . In particular, one can show that for  $|h| \geq 10$  the relative error  $\epsilon$  between the approximation of Eq. (16) and the result obtained from the annealed approximation vanishes  $\sim |h|^{-1}$  (inset of Fig. 6), i.e. it becomes *exact* for  $|h| \rightarrow \infty$ . In a double-logarithmic plot, the solution curve  $K_c(|h|)$  appears to approach a straight line for large  $|h|$ , suggesting that  $K_c(|h|)$  converges to a power-law  $K_c(|h|) = a \cdot |h|^\alpha$ . We confirmed this intuition by numerically inserting candidate solutions with fixed  $\alpha$  into Eq. (15), and solving for the values of  $|h|$  and  $a$  where the deviation from the true curve  $K_c(|h|)$  becomes minimal; inverting this relation, we obtain the optimal power law exponents  $\alpha(|h|)$  as a function of  $|h|$  (Fig. 7, for details, see appendix C). One can show that  $\alpha(|h|)$  is fit excellently by

$$\alpha(|h|) \approx \alpha_\infty - c_1 \cdot [\ln |h|]^{-\beta_1} \quad (17)$$

with  $\alpha_\infty = 1.97821 \pm 0.004$ ,  $c_1 = 1.88681 \pm 0.06$  and  $\beta_1 = 1.32056 \pm 0.03244$ ; similarly, we find

$$a(|h|) \approx a_\infty + c_2 \cdot [\ln |h|]^{-\beta_2} \quad (18)$$

with  $a_\infty = 0.06288 \pm 0.009$ ,  $c_2 = 6.70426 \pm 0.354$  and  $\beta_2 = 1.75106 \pm 0.048$ . Hence,  $K_c(|h|)$  indeed logarithmically approaches an asymptotic, unique power law for large  $|h|$ :

$$\lim_{|h| \rightarrow \infty} K_c(|h|) = 0.06288 \cdot |h|^{1.97821}. \quad (19)$$

So far, we did not find an exact analytical proof for this relationship, due to the intricate, implicit dependence between  $|h|$  and  $\bar{K}$  in the transcendental Eq. (15), however, one can make it plausible that the scaling relationship should approach a power-law with an exponent at the order of 2 asymptotically. For this purpose, let us first study the scaling behavior of the maximum of  $p_s(k, |h|)$  with respect to  $|h|$ ; if we restrict our analysis to even  $k - |h|$ ,  $k_{max}$  is given by the condition

$$\Delta p_s = p_s(k, |h|) - p_s(k-2, |h|) \approx 0 \quad (20)$$

or, more accurately, we have to find the minimum of the absolute value  $|\Delta p_s / \Delta k|$  of the 'discrete derivative' of  $p_s(k, |h|)$  for even  $k - |h|$ , with  $\Delta k = \text{const.} = 2$ . Inserting Eq. (8) then yields

$$\Delta p_s = 2^{-k+3} \frac{(k-3)!}{[(k+|h|-3)/2]![(k-|h|-3)/2]!} \cdot \left\{ \frac{(k-1)(k-2)}{(k+|h|+1)(k-|h|-1)} - 1 \right\}. \quad (21)$$

Obviously, the pre-factor on the right hand-side is always positive; consequently, in order to determine the maximum of  $p_s(k, |h|)$ , we have to solve the equation

$$\frac{(k-1)(k-2)}{(k+|h|+1)(k-|h|-1)} - 1 = 0. \quad (22)$$

Using simple algebra, one can show that

$$k_{max} = |h|^2 + 1 \quad (23)$$

solves this equation, i.e. the maximum of  $p_s(k, |h|)$  scales quadratically with  $|h|$ . Since  $p_s(k, |h|)$  for  $|h| \gg 0$  vanishes both for small and large  $k$ , it is plausible that the scaling behavior of  $K_c$  is dominated by the leading behavior of the maximum of the distribution, i.e. should scale  $\sim |h|^\alpha$  with an exponent  $\alpha$  at the order of 2.

Figure 8 shows that, for finite  $|h|$ , the critical line  $K_c(|h|)$  for RTN with inhomogeneous thresholds is always *below* the corresponding values for homogeneous  $|h|$ ; the absolute difference  $\Delta K_c(|h|) := |K_c^h(|h|) - K_c^i(|h|)|$  between both curves, however, increases only linearly in with  $|h|$  (inset of Fig. 8), where  $K_c^h(|h|)$  is the critical connectivity for homogeneous  $|h|$ , and  $K_c^i(|h|)$  the corresponding value for inhomogeneously distributed  $|h|$  with mean  $|\bar{h}| = |h|$ .

This implies that, for  $|h| \rightarrow \infty$ , both scaling functions converge to the same asymptotic scaling relationship

$$K_c(|h|) \propto |h|^\alpha \quad (24)$$

with the associated scaling exponent

$$\alpha = 1.97821 \pm 0.004. \quad (25)$$

Intuitively, this is straight-forward to understand: since we assumed that  $k$  and  $|h|$  are *statistically independent*,  $\Delta K_c(|h|)$  is determined solely by the *variance*  $\text{Var}(|h|)$  of the threshold distribution around the mean threshold  $|\bar{h}| = |h|$  - the smaller this variance is, the more peaked this distribution is around  $|\bar{h}| = |h|$ , and the less it hence differs from the homogeneous distribution. Since we assumed that (in the inhomogeneous case) thresholds are Poisson distributed around  $|\bar{h}|$ , we directly conclude

$$\Delta K_c(|h|) \sim \text{Var}(|h|) = |\bar{h}|, \quad (26)$$

consequently, the asymptotic scaling is determined by the dominating power  $|h|^\alpha$  with  $\alpha \approx 2$  (this will be discussed in more detail below). This observation now gives us an intuition for a further extension of this universality class: we expect that Eq. 24 with the associated scaling exponent (Eq. 25) is universal for all networks with Poisson distributed connectivity and a threshold distribution with a variance that grows slower than  $|\bar{h}|^\alpha$ . In all these cases, the asymptotic scaling for  $|\bar{h}| \rightarrow \infty$  is dominated by the scaling behavior of the maximum of the damage propagation function  $p_s(k, |h|)$ , with an exponent at the order of 2.

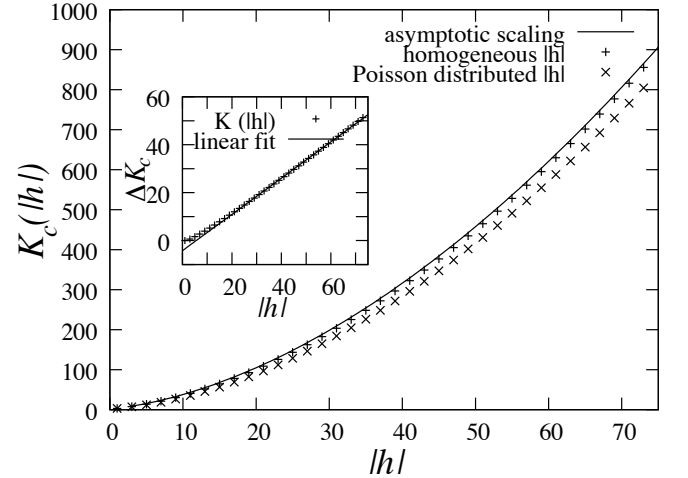


FIG. 8:  $K_c(|h|)$  for homogeneous thresholds (+) and Poisson distributed thresholds with the same *average*  $|\bar{h}|$  (X), annealed approximation. The solid line is the asymptotic scaling obtained from Eq. (16). For inhomogeneous  $|h|$ , the critical line is systematically below  $K_c$  of networks with homogeneous  $|h|$ . *Inset:* The difference  $|\Delta K_c(|h|)|$  between both curves grows only linearly in  $|h|$ , confirming that the asymptotic scaling in the limit  $|h| \rightarrow \infty$ , is the same in both cases.

Let us now test this conjecture for a different class of threshold distributions. Since in a Poissonian the variance is not a free parameter, we now instead choose a discretized Gaussian distribution, i.e.

$$P(|h|) = \frac{Z}{\sigma_h \sqrt{2\pi}} e^{-\frac{1}{2}(|h| - |\bar{h}|)^2 / \sigma_h^2} \quad (27)$$

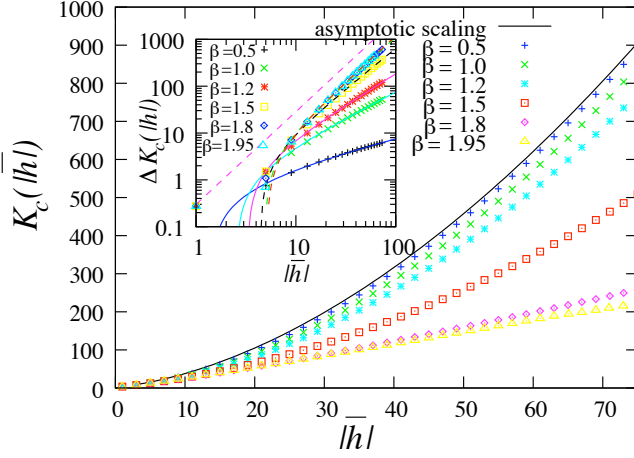


FIG. 9:  $K_c(\beta, |\bar{h}|)$  for networks with threshold distributions following discretized Gaussian distributions with different variances  $\text{Var}(|h|) = |\bar{h}|^\beta$  (for details, see text). One clearly appreciates that the larger the variance of the threshold distribution, the more the curves  $K_c(\beta, |\bar{h}|)$  are below the critical line of networks with homogeneous thresholds (blue solid line); in the limiting case  $\beta = 1.95 \approx \alpha$  (yellow triangles),  $K_c$  scales almost linearly with  $|\bar{h}|$ . *Inset:* differences  $|\Delta K_c(\beta, |\bar{h}|)|$  to the critical line of RTN with homogeneous thresholds scale  $\sim |\bar{h}|^{\beta_e}$  with  $\beta < \beta_e < \alpha$  (power law fits and dashed line with slope  $\alpha$  shown for comparison); this implies asymptotic convergence to the universal scaling function Eq. (24) in the limit  $|h| \rightarrow \infty$  for all cases shown here.

$\beta$	$\beta_e$
0.5	$0.533 \pm 0.009$
1.0	$1.099 \pm 0.004$
1.2	$1.327 \pm 0.004$
1.5	$1.732 \pm 0.004$
1.8	$1.942 \pm 0.003$
1.95	$1.975 \pm 0.004$

TABLE I: Scaling exponents  $\beta_e$ , as obtained from fits of  $\Delta K_c \sim |\bar{h}|^{\beta_e}$ , as a function of  $\beta$ .

with

$$Z = \left\{ \sum_{|h|=0}^{|h|_m} \frac{1}{\sigma_h \sqrt{2\pi}} e^{-\frac{1}{2}(|h| - |\bar{h}|)^2 / \sigma_h^2} \right\}^{-1} \quad (28)$$

and variance

$$\sigma_h^2 = |\bar{h}|^\beta, \quad \beta \in [0, \alpha]. \quad (29)$$

The factor  $Z$  ensures that the probabilities are normalized in the interval  $[0, |h|_m]$ , where  $|h|_m$  denotes the cut-off of the threshold distribution. Figure 9 compares the scaling functions  $K_c(|\bar{h}|)$  for different values of  $\beta$  to the asymptotic case of homogeneous networks. Obviously, for finite  $|\bar{h}|$ , increased variance of the threshold distribution substantially lowers the critical connectivity; in

the limiting case  $\beta \approx \alpha$ ,  $K_c$  grows only linearly with  $|\bar{h}|$ . For  $\beta < \alpha$ , we find that the deviation from the scaling behavior of RTN with homogeneous thresholds scales as

$$\Delta K_c \propto |\bar{h}|^{\beta_e}. \quad (30)$$

Table 1 compares  $\beta$  and  $\beta_e$  (as obtained from fits of  $\Delta K_c$ ; in all cases, we have  $\beta_e > \beta$ , which is a discretization effect, but still  $\beta_e < \alpha$ ). Hence, it follows that

$$\begin{aligned} \lim_{|h| \rightarrow \infty} \frac{K_c(\beta, |\bar{h}| = |h|)}{K_c^h(|h|)} &= \lim_{|h| \rightarrow \infty} \frac{K_c^h(|h|) - \Delta K_c(\beta, |h|)}{K_c^h(|h|)} \\ &= 1 - \text{const.} \cdot \lim_{|h| \rightarrow \infty} |h|^{\beta_e - \alpha} \\ &= 1 \end{aligned} \quad (31)$$

for  $\beta_e < \alpha$ , i.e. in this case all scaling functions  $K_c(\beta, |\bar{h}|)$  for  $|h| \rightarrow \infty$  asymptotically converge to the same universal scaling function, as given by Eq. (24).

Finally, let us have a closer look at the scaling behavior of the intersection points  $K_d(|h|)$ , as introduced in the last paragraph of subsection B. Let  $\bar{d}^h(\bar{K}, |h|)$  be the expected damage in networks with homogeneous threshold, and  $\bar{d}^i(\bar{K}, |h|)$  the expected damage in networks with inhomogeneous thresholds; then

$$\bar{d}^h(K_d(|h|), |h|) - \bar{d}^i(K_d(|h|), |\bar{h}|) = 0 \quad (32)$$

$$|\bar{h}| = |h| \quad (33)$$

are the defining equations for  $K_d(|h|)$ . Notice that for  $\bar{K} < K_d$ , the randomness introduced by inhomogeneous thresholds actually *increases* the probability for damage spreading, whereas for  $\bar{K} > K_d$ , it is *decreased*. Equation (32), under condition Eq. (33), can be solved numerically for not too large  $|h|$ ; Fig. 10 demonstrates that  $K_d(|h|)$  approaches the asymptotic scaling  $K_d(|h|) \sim |h|^\alpha$  already for considerably small  $|h|$ , indicating that  $K_d(|h|)$  is characterized by the same scaling exponent  $\alpha$  as  $K_c(|h|)$ .

It is interesting to notice that the dependence of  $K_c$ , as well as of  $K_d$  on  $|h|$  is clearly super-linear even for considerably small  $|h|$ ; this has profound consequences for algorithms that evolve RTN towards (self-organized) criticality by local adaptations of both thresholds and the number of inputs a node receives from other nodes [31]. In particular, it can be shown that co-evolution of network dynamics and thresholds/in-degrees leads to strong correlations between  $|h|$  and  $k$ , and broad in-degree distributions. To approach this type of problem analytically, we will now extend our analysis in two more steps: first, we will study damage spreading for networks with broad in-degree distributions (in particular, with a power-law tail), second, we will show that even weak correlations between  $k$  and  $|h|$  can lead to a transition from sub-critical to super-critical dynamics (and vice versa), while keeping the average connectivity  $\bar{K}$  and the average absolute threshold  $|\bar{h}|$  constant.



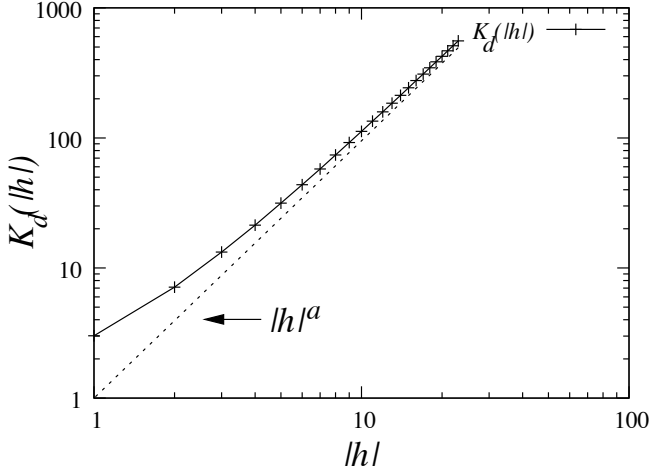


FIG. 10: Scaling behavior of  $K_d(|h|)$  as a function of  $|h|$ , double logarithmic plot. A straight dashed line with slope  $\alpha = 1.97844$  is shown for comparison.

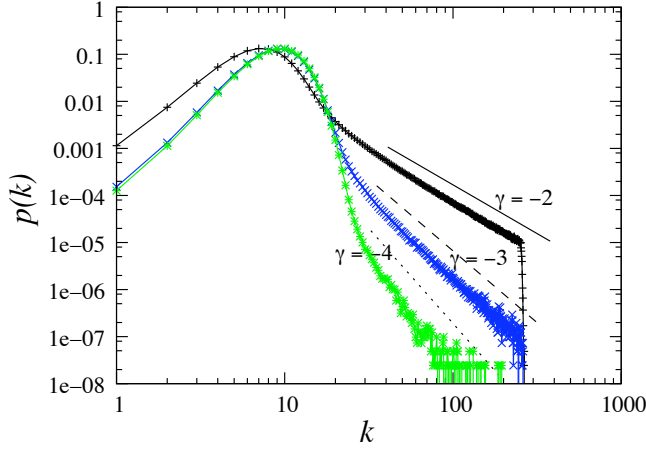


FIG. 11: Probability distributions  $p(k)$  of the in-degree  $k$  of random networks generated according to the algorithm described in Appendix D, for  $\bar{K} = 10$ , averaged over ensembles of  $10^4$  networks for each distribution. Notice the power-law tails of the distributions (lines with slope  $-2$ ,  $-3$  and  $-4$  shown for comparison), while for small  $k$ , the distributions resemble a Poissonian. The cutoff of the distributions was fixed to  $k_m = 250$ .

#### D. Scale-free topologies

Let us now consider random networks with broader in-degree distributions - in particular, scale-free networks, i.e. networks with

$$\rho_{k_{in}} \propto k^{-\gamma}. \quad (34)$$

Dynamics of RTN with power-law distributed in-degree was already investigated in [24]; in that model, the connectivity was assumed to have a cutoff at a considerably small  $k = k_m \approx 100$ , with  $\bar{K}$  being a function of both

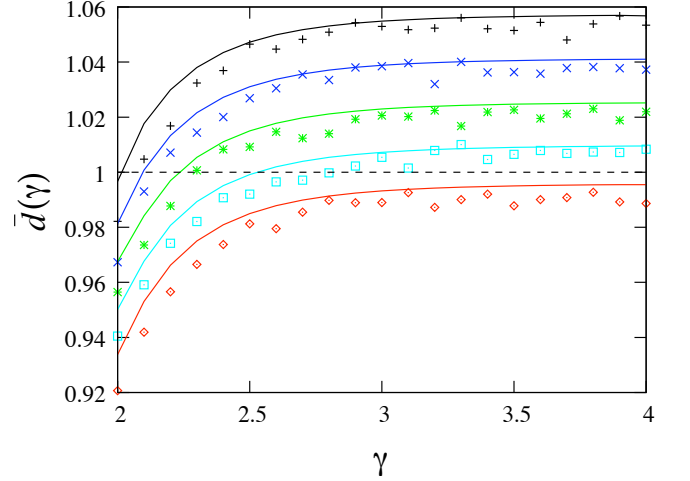


FIG. 12: Average damage  $\bar{d}(\gamma)$  as a function of the power law exponent  $\gamma$  of the corresponding in-degree distributions (cf. Fig. 11), ensemble average over  $10^4$  randomly generated networks with  $N = 4096$ ,  $\bar{K} = 10$  and  $k_m = 250$  for each data point, and Poissonian distributed thresholds with average threshold  $|\bar{h}|$  as a free parameter; the corresponding solid curves are calculated from the annealed approximation. From top to bottom:  $|\bar{h}| = 4.0$ ,  $|\bar{h}| = 4.05$ ,  $|\bar{h}| = 4.1$ ,  $|\bar{h}| = 4.15$  and  $|\bar{h}| = 4.2$ . Notice that transitions from subcritical (ordered) to supercritical (chaotic) networks are found only for  $|\bar{h}| < 4.2$ .

the fraction  $D$  of disconnected sites and the scaling exponent  $\gamma$ . In this setting, it is not possible to fix  $\bar{K}$ ,  $\gamma$  and  $k_m$  at the same time. However, due to their finite size, the degree-distributions of most real-world networks show only scale-free tails, while  $\rho_{k_{in}}$  for small  $k$  can exhibit considerable deviations from the power-law dependence. In Appendix D, we describe an algorithm that is capable to generate ensembles of finite-size networks for fixed  $\bar{K}$ ,  $\gamma$  and  $k_m$ ; while the in-degree distributions of these networks have scale-free tails  $\propto k^{-\gamma}$ , for small  $k$  the distributions rather resemble a Poissonian (Fig. 11). Fig. 12 shows the average damage  $\bar{d}(\gamma)$  as a function of the power law exponent  $\gamma$ , for different values of the average threshold  $|\bar{h}|$ . One can clearly appreciate that for smaller  $\gamma$  (corresponding to flat power law tails) damage is significantly reduced, leading to a transition from supercritical (chaotic) to subcritical (ordered) dynamics, as indicated by the crossing of the line  $\bar{d} = 1$ . Similar results were found in earlier studies [23, 24], however, without being able to distinguish the effects of increasing network disconnection with decreasing  $\gamma$  (and constant  $\bar{K}$ ) and damage suppression at network hubs. Since in the network connection algorithm discussed in appendix D, disconnection of network sites is avoided (cf. also Fig. 11: the probability of sparsely connected sites is rather decreased than increased for smaller  $\gamma$ ), the transition discussed here evidently is an effect of damage suppression at in-degree hubs, as suggested by the asymptotic decay of the damage propagation probability  $p(k, |h|)$  for

large  $k$  (Fig. 1). Notice that the numerically measured damage values for  $\gamma \rightarrow 2$  are systematically below the prediction of the annealed approximation, indicating considerable sample-to-sample fluctuations of connectivity patterns in the randomly generated finite size RTN; for large  $N$ , however, these fluctuations vanish. It is interesting to notice that the observed damage suppression effect in Fig. 12 for inhomogeneous  $|\bar{h}| \approx 4$  is stronger than the effect described for  $|h| = 0$  in [23] and for constant non-zero  $|h|$  in [24] and occurs already for  $\gamma \approx 2.5$ , indicating that coincidence of non-trivial network topology *and* inhomogeneous thresholds has a considerable order-inducing effect, even if both are uncorrelated.

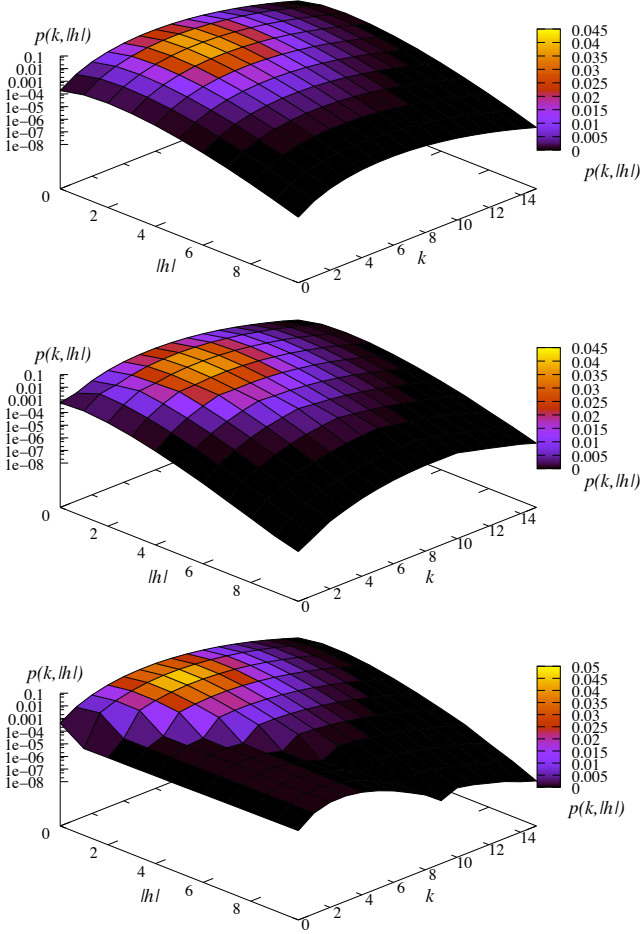


FIG. 13: Correlation matrices  $p_c(k, |h|)$  for  $c = 0$  (no correlations between  $k$  and  $|h|$ , upper panel), correlations between  $k$  and  $|h|$  according to a correlation parameter value  $c = 0.95$  (middle panel) and anti-correlations according to  $c = 0.95$  (lower-panel), averaged over network ensembles of  $Z = 5 \cdot 10^5$  RTN randomly generated according to the algorithm described in the text for each case.

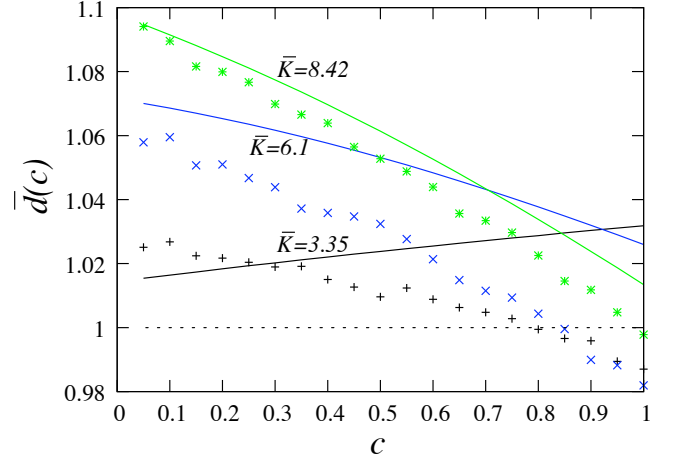


FIG. 14: Average damage  $\bar{d}(c)$  as a function of  $c$ , for correlated  $k$  and  $|h|$ , with  $\bar{K}$  and  $|\bar{h}|$  chosen slightly *above* the critical values for uncorrelated networks. Data were obtained from ensemble averages over  $Z = 5 \cdot 10^5$  randomly generated RTN with  $N = 1024$  nodes for each data point. Solid curves with the same color are the corresponding results of the annealed approximation.

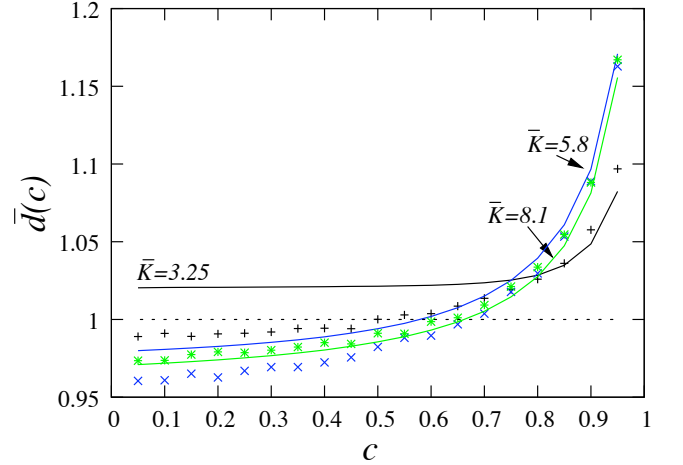


FIG. 15: Average damage  $\bar{d}(c)$  as a function of  $c$ , for anti-correlated  $k$  and  $|h|$ , with  $\bar{K}$  and  $|\bar{h}|$  chosen slightly *below* the critical values for uncorrelated networks. Data were obtained from ensemble averages over  $Z = 5 \cdot 10^5$  randomly generated RTN with  $N = 1024$  nodes for each data point. Solid curves with the same color are the corresponding results of the annealed approximation.

### E. Effect of correlations between $k$ and $h$

So far, we assumed that node degree and node thresholds are totally uncorrelated; while this matches well the “maximum disorder” assumption used in random ensemble based approaches as, e.g., the annealed approximation, this might be a quite unrealistic constraint for many real world networks. Indeed, one can show that even in

a simple evolutionary algorithm that couples both the adaptation of node thresholds  $h_i$  and in-degree  $k_i$  to a local dynamical order parameter, strong correlations between both quantities emerge spontaneously [31]. Hence, it is an interesting question to ask whether correlations (or anti-correlations) between  $h$  and  $k$  may induce a transition from sub-critical to super-critical networks (or vice versa), while we keep  $\bar{K}$  and  $|\bar{h}|$ , and network topologies constant.

Let us first introduce an algorithm that generates weak correlations (anti-correlations) between  $k$  and  $h$ :

- 1) Generate a random, directed network with Poisson distributed  $k$ , average connectivity  $\bar{K}$  and  $h = 0$  for all sites.
- 2) Select a site  $i$  at random.
- 3) With probability  $c$ , decrement  $h_i \rightarrow h_i - 1$  if  $k_i > |h_i|$  ( $k_i < |h_i|$ ); with probability  $1 - c$ , decrement  $h_i \rightarrow h_i - 1$  regardless of  $k_i$ .
- 4) Go back to 2) and repeat, until the network's average absolute threshold has reached the desired value  $|\bar{h}|$ .

Obviously, increasing the parameter  $c \in [0, 1]$  increases correlations (anti-correlations) between  $k$  and  $h$ . If we repeat this algorithm  $Z$  times for fixed  $c$ , we can generate a random ensemble of  $Z$  correlated/anti-correlated networks, and investigate damage spreading on these networks. The ensemble-averaged probability  $p_c(k, |h|)$  to have a site with  $k$  inputs and threshold  $|h_i| = |h|$  then is defined as

$$p_c(k, |h|) = \frac{\sum_{j=1}^Z n_j(k, |h|)}{Z \cdot N}, \quad (35)$$

where  $n_j(k, |h|)$  is the number of sites with  $k$  inputs and threshold  $|h_i| = |h|$  in the  $j$ th random network. Figure 13 demonstrates the correlating effect of the algorithm on the *average* probabilities  $p(k, |h|)$  for ensembles of 500000 randomly generated networks. Notice that, for positive correlations, the effect is considerably small even for  $c = 0.95$  (middle panel), while it is considerably larger for anti-correlations (bottom panel); presumably, the limited size of correlations is due to the small variance of the (Poisson-distributed) degree, while anti-correlations more effectively exploit the asymmetry of the degree-distribution.

Let us now investigate how these correlations affect damage propagation. To apply the annealed approximation, we now have to calculate the average probability for damage propagation (in a finite network of size  $N$ ) according to

$$\langle p_s \rangle(\bar{K}, |\bar{h}|, c) = \sum_{|h|=0}^{|\bar{h}|_m} \sum_{k=|h|}^N p_c(k, |h|) p_s(k+1, |h|), \quad (36)$$

with the normalization conditions

$$\sum_{|h|=0}^{|\bar{h}|_m} \sum_{k=|h|}^N p_c(k, |h|) = 1 \quad (37)$$

and

$$\sum_{|h|=0}^{|\bar{h}|_m} \sum_{k=|h|}^N |h| p_c(k, |h|) = |\bar{h}|, \quad (38)$$

where  $|\bar{h}|_m$  is the maximal absolute threshold observed (cutoff); correlations enter via the probabilities  $p_c(k, |h|)$  to observe a node with degree  $k$  and absolute threshold  $|h|$ . Notice, however, that these probabilities are averaged over the network ensemble, and therefore only approximate the true damage propagation behavior for a specific network. Interestingly, for sparse (small)  $\bar{K}$  and average threshold values slightly above the critical value, in numerical simulations of RTN dynamics we find a systematic *decrease* of the average damage  $\bar{d}$  with increasing correlation parameter  $c$ , while the annealed approximation predicts an *increase* ( $\bar{K} = 3.35$  in Fig. 14), while for larger  $\bar{K}$  the annealed approximation better matches numerical data; similar systematic deviations are found for the case of anti-correlations (Fig. 15). Still, future studies may show that the deviations vanish in the limit  $N \rightarrow \infty$ , notice, however, that the annealed approximation as a *mean-field theory* systematically neglects all topological and dynamical correlations. Hence, we believe that new theoretic approaches are needed to describe the limiting cases of sparsely connected RTN with correlated  $k$  and  $|h|$ , that go beyond mean-field approximations.

#### IV. DISCUSSION

We studied damage propagation in Random Threshold Networks (RTN) with homogeneous and inhomogeneous negative thresholds, both analytically (using an annealed approximation) and in numerical simulations. We derived the probability  $p_s(k, |h|)$  of damage propagation for arbitrary in-degree  $k$  and (absolute) threshold  $|h|$  (Eqs. (5)-(8)), and, from this, the corresponding *annealed* probabilities  $\langle p_s \rangle$  (Eq. (10) and Eq. (12)) and the expected damage  $\bar{d}$  (Eq. (11)), for both the cases of homogeneous and inhomogeneously distributed thresholds. Interestingly, inhomogeneity in thresholds, meaning that each site has an individual threshold  $|h_i|$  drawn, e.g., from a Poisson distribution with mean  $|\bar{h}|$ , increases damage for small average connectivity  $\bar{K}$ , when compared to homogeneous networks with the same average threshold  $|\bar{h}| = \bar{h}$ , whereas for larger  $\bar{K}$  with  $\bar{K} > K_d$ , damage is reduced. This establishes a new characteristic connectivity  $K_d(|h|)$  with  $K_d > K_c$ , that describes the ambivalent effect of threshold inhomogeneity on RTN dynamics.

Further, we investigated the scaling behavior of the critical connectivity  $K_c$  as a function of  $|h|$ . Using a mean field approximation, a simplified scaling equation for the logarithm of the average damage was derived (Eq. (14)), and applied to derive the critical line  $K_c(|h|)$  (Fig. 6). It was shown that this function exhibits a super-linear increase with  $|h|$ , which asymptotically approaches

a unique power-law  $K_c(|h|) \sim |h|^\alpha$  with  $\alpha \approx 2$  for large  $|h|$  (Eq. (18) and Fig. 7). We presented evidence that this asymptotic scaling is universal for RTN with Poissonian distributed connectivity and threshold distributions with a variance that grows slower than  $|h|^\alpha$ , for both the cases of Poisson distributed thresholds (Fig. 8) and thresholds distributed according to a discretized Gaussian (Fig. 9).

Next, we studied damage propagation in RTN with in-degree distributions that exhibit a power-law tail  $\sim k^\gamma$ , while keeping all other network parameters constant (namely, the average connectivity  $\bar{K}$  and average threshold  $|\bar{h}|$ ; our results showed that the coincidence of non-trivial network topology and inhomogeneous thresholds has a considerable order-inducing effect for  $\gamma \rightarrow 2$  (Fig. 12), which is clearly an effect of damage suppression in in-degree "hubs", which becomes prominent for small  $\gamma$ . Last, we showed that even weak (anti-)correlations between  $k$  and  $|h|$  can induce an order-disorder transition (or vice versa, cf. Fig. 14 and 15); interestingly, for sparse  $\bar{K}$  the annealed approximation fails to predict both size and sign of the change in damage propagation induced by the correlations, pointing at the limits of this theory when applied to non-trivial network topologies.

To summarize, dynamics of damage (or information) propagation in RTN with inhomogeneous thresholds and Poisson distributed connectivity shows both similarities

and differences, when compared to networks with homogeneous thresholds: similarities manifest themselves in common universal scaling functions for both  $K_c$  and  $K_d$ , whereas differences show up in the opposite effects of threshold inhomogeneity for small and large  $\bar{K}$ . Differences become even more prominent in networks that are characterized by scale-free in-degree distributions, or correlations between in-degree and thresholds. Many dynamical systems in nature, that can be described as complex networks, exhibit considerable variation of activation thresholds among the elements they consist of, however, these variations are often neglected (e.g., in Boolean network based models of gene regulation networks). Our results indicate that, while general characteristics as, for example, the scaling behavior of critical points, may be conserved in approximations of this type, inhomogeneous thresholds can strongly impact the details of network dynamics, and hence should be taken into account in models that aim to give a realistic description of the dynamics of, e.g., gene regulation networks.

## V. ACKNOWLEDGMENTS

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  - [32] We restrict ourselves to negative (or zero) thresholds, to ensure that the 'default state' of a network site  $i$ , i.e. when its inputs sum to zero, is to be 'inactive' ( $\sigma_i = -1$ ), which naturally excludes positive thresholds.
  - [33] Other authors define  $\text{sgn}(0) = +1$ , however, for symmetry reasons update dynamics is not affected by either choice. If we interpret the state  $\sigma_i = -1$  as 'inactive' and,

correspondingly, +1 as 'active', our choice appears to be more natural: the default state of a network site is to be 'inactive', unless it receives activating inputs from other sites.

[34] To obtain accurate results, one has to consider networks sizes  $N \gg \bar{K}$ , and adjust the upper limit of the sum in (10) accordingly. Since a small step size  $\Delta \bar{K}$  has to be applied iteratedly to identify  $K_c$ , this becomes computationally very costly.

## APPENDIX A: DERIVATION OF $p_s(k, |h|)$

In this section, we provide a derivation of the local damage propagation probability  $p_s(k, |h|)$ .

Consider a network site  $i$  with  $k$  inputs;  $k_+$  of these have positive sign,  $k_-$  negative sign, hence,  $k_+ + k_- = k$ . We now derive the conditions under which an inversion of one input spin at time  $t$  leads to a switch of the output of site  $i$  at time  $t + 1$ .

**1)  $k - |h|$  odd:** From Eqs. 1 and 2 it is easy to see that input-spin flips produce "damage" only if one of the following conditions holds:

$$k_+ - k_- - |h| = 1 \quad (\text{A1})$$

or

$$k_+ - k_- - |h| = -1. \quad (\text{A2})$$

In case A1, only the reversal of positive spins is effective, whereas in case A2, only the reversal of negative spins has an effect. We have

$$k_+ = \frac{k + |h| + 1}{2} \quad (\text{A3})$$

in the first case and

$$k_- = \frac{k - |h| + 1}{2} \quad (\text{A4})$$

in the second case. There is a total number of  $k \cdot 2^k$  possible spin configurations, of which  $\binom{k}{(k+|h|+1)/2}$  fulfill condition A3 and  $\binom{k}{(k-|h|+1)/2}$  fulfill condition A4. Hence, the damage propagation probability follows as

$$p_s(k, |h|) = k^{-1} \cdot 2^{-(k-1)} \cdot \left[ (k + |h| + 1) \cdot \left( \frac{k}{\frac{k+|h|+1}{2}} \right) + (k - |h| + 1) \cdot \left( \frac{k}{\frac{k-|h|+1}{2}} \right) \right] \quad (\text{A5})$$

$$= \frac{2^{-(k-1)} (k-1)!}{[(k + |h| - 1)/2]! [(k - |h| - 1)/2]!} \quad (\text{A6})$$

$$= 2^{-(k-1)} \binom{k-1}{\frac{k+|h|-1}{2}}. \quad (\text{A7})$$

**2)  $k - |h|$  even:** Here, we have as necessary conditions

$$k_+ - k_- - |h| = 0 \quad (\text{A8})$$

or

$$k_+ - k_- - |h| = 2. \quad (\text{A9})$$

In the first case, only the reversal of negative spins is effective, whereas in the latter case the same holds for positive spins. We have

$$k_- = \frac{k - |h|}{2} \quad (\text{A10})$$

in the first case and

$$k_+ = \frac{k + |h| + 2}{2} \quad (\text{A11})$$

in the second case. There is a total number of  $k \cdot 2^k$  possible spin configurations, of which  $\binom{k}{(k-|h|)/2}$  fulfill condition A10 and  $\binom{k}{(k+|h|+2)/2}$  fulfill condition A11. Hence, the damage propagation probability follows as

$$p_s(k, |h|) = k^{-1} \cdot 2^{-(k-1)} \cdot \left[ (k - |h|) \cdot \left( \frac{k}{\frac{k-|h|}{2}} \right) + (k + |h| + 2) \cdot \left( \frac{k}{\frac{k+|h|+2}{2}} \right) \right] \quad (\text{A12})$$

$$= \frac{2^{-(k-1)} (k-1)!}{[(k - |h| - 2)/2]! [(k + |h|)/2]!} \quad (\text{A13})$$

$$= 2^{-(k-1)} \binom{k-1}{\frac{k+|h|}{2}}. \quad (\text{A14})$$

## APPENDIX B: DERIVATION OF THE SCALING EQUATION

For RTN with Poisson distributed in- and out-degree, the critical line is given by the condition

$$\bar{d}(t+1) = \langle p_s \rangle (K_c(|h|), |h|) \cdot K_c(|h|) = 1. \quad (\text{B1})$$

with

$$\langle p_s \rangle (\bar{K}, |h|) = e^{-\bar{K}} \sum_{k=|h|}^{\bar{K}} \frac{\bar{K}^k}{k!} p_s(k+1, |h|). \quad (\text{B2})$$

Instead of averaging over the ensemble of all possible network topologies as in Eq. (B2), we now make an explicit *mean field approximation*, and consider a "typical" network node with  $k \approx \bar{K}$  inputs. Consequently, we approximate

$$\langle p_s \rangle (\bar{K}, |h|) \approx p_s(\lfloor \bar{K} \rfloor, |h|), \quad (\text{B3})$$

where  $\lfloor \cdot \rfloor$  denotes the floor function. In the limit of large  $\bar{K}$  and  $|h|$ , the difference between the damage propagation probabilities for even and odd  $k$  vanishes, i.e. we can set

$$\langle p_s \rangle (\bar{K}, |h|) \approx 2^{-(\lfloor \bar{K} \rfloor - 1)} \binom{\lfloor \bar{K} \rfloor - 1}{\frac{\lfloor \bar{K} \rfloor + |h|}{2}}, \quad (\text{B4})$$

and hence

$$\bar{d}(\bar{K}, |h|) = \bar{K} \cdot 2^{-\lfloor \bar{K} \rfloor} \left( \frac{\lfloor \bar{K} \rfloor}{\frac{\lfloor \bar{K} \rfloor + |h|}{2}} \right) \quad (\text{B5})$$

without loss of generality.

Using the Stirling approximation

$$n! \approx n^n e^{-n} \sqrt{2\pi n}, \quad (\text{B6})$$

dropping the floor function (since we now consider a function of real-valued variables only) and taking logarithms, we obtain

$$\ln [\bar{d}(\bar{K}, |h|)] \approx \ln \bar{K} + Z_1 - Z_2 - Z_3 \quad (\text{B7})$$

with

$$Z_1 = \ln [\bar{K}^{\bar{K}} e^{-\bar{K}} \sqrt{2\pi \bar{K}}],$$

$$Z_2 = \ln \left[ \left( \frac{\bar{K} - |h|}{2} \right)^{\frac{\bar{K} - |h|}{2}} e^{-\frac{\bar{K} - |h|}{2}} \sqrt{\pi(\bar{K} - |h|)} \right]$$

and

$$Z_3 = \ln \left[ \left( \frac{\bar{K} + |h|}{2} \right)^{\frac{\bar{K} + |h|}{2}} e^{-\frac{\bar{K} + |h|}{2}} \sqrt{\pi(\bar{K} + |h|)} \right]$$

Summing out the logarithms in  $Z_1$ ,  $Z_2$  and  $Z_3$ , one realizes that all terms linear in  $\bar{K}$  drop out, resulting in

$$\begin{aligned} \ln [\bar{d}(\bar{K}, |h|)] &\approx \ln \bar{K} + \left( \bar{K} - \frac{1}{2} \right) \ln \bar{K} \\ &\quad - \frac{\bar{K} - |h| + 1}{2} \ln (\bar{K} - |h|) \\ &\quad - \frac{\bar{K} + |h| + 1}{2} \ln (\bar{K} + |h|) + C \end{aligned} \quad (\text{B8})$$

with  $C = \ln(\sqrt{2/\pi})$ . Using some simple algebra and approximating  $|h| + 1 \approx |h|$ , this can be reformulated as

$$\begin{aligned} \ln [\bar{d}(\bar{K}, |h|)] &\approx \ln \bar{K} - \frac{1}{2} \left\{ \ln (\bar{K} \right. \\ &\quad \left. - \bar{K} \ln \left[ \frac{(\bar{K} + |h|)(\bar{K} - |h|)}{\bar{K}^2} \right] \right. \\ &\quad \left. + |h| \ln \left[ \frac{\bar{K} + |h|}{\bar{K} - |h|} \right] \right\} + C. \end{aligned} \quad (\text{B9})$$

This leads to the final result

$$\begin{aligned} \ln [\bar{d}(\bar{K}, |h|)] &\approx \frac{1}{2} \left\{ \ln \bar{K} - \bar{K} \cdot \ln \left[ 1 - \left( \frac{|h|}{\bar{K}} \right)^2 \right] \right. \\ &\quad \left. - |h| \ln \left[ \frac{\bar{K} + |h|}{\bar{K} - |h|} \right] \right\} + C. \end{aligned} \quad (\text{B10})$$

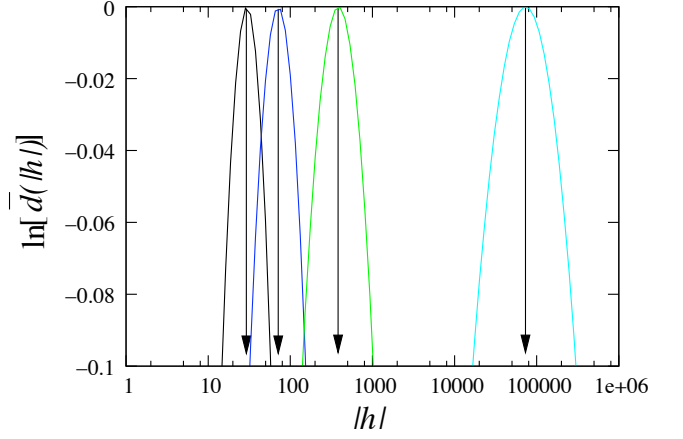


FIG. 16: Solutions of Eq. (C3) for (from the left to the right)  $\alpha = 1.6$ ,  $\alpha = 1.7$ ,  $\alpha = 1.8$  and  $\alpha = 1.9$ . Projections of the maximum on the  $|h|$ -axis (as indicated by arrows) yield the corresponding values of  $|h|_c$  at which the approximations are optimal.

### APPENDIX C: POWER-LAW APPROXIMATION OF $K_c(|h|)$ FOR FINITE $|h|$

In this section we describe how to identify candidate solutions (power-laws)

$$K_c(|h|) \approx a(|h|) \cdot |h|^{\alpha(|h|)} \quad (\text{C1})$$

that optimally approximate Eq. (15) for finite (critical)  $|h|_c$ .

We start with a fixed  $\alpha \in [1.6, 2)$  and define

$$\begin{aligned} F(y) &:= \frac{1}{2} \left\{ \ln y - y \cdot \ln \left[ 1 - \left( \frac{|h|}{y} \right)^2 \right] \right. \\ &\quad \left. - (|h| + 1) \ln \left[ \frac{y + |h|}{y - |h|} \right] \right\} + C \end{aligned} \quad (\text{C2})$$

with  $y = a \cdot |h|^\alpha$ . One can show that, for any finite  $a$  and  $\alpha$ ,  $F(y)$  has a maximum at a finite value  $|h|_{max}$ . We know that  $K_c$  is a monotonically increasing function of  $|h|$ , and intend to optimize the power-law approximation exactly at  $K_c$ . Hence, we have to vary  $a$  such that

$$\max_a F(y)|_\alpha = 0. \quad (\text{C3})$$

Projection of the maximum on the  $|h|$ -axis then yields the corresponding threshold values  $|h|_c(\alpha)$  at which the approximation for the given  $\alpha$  is optimal (Fig. 16). Inversion of this relation allows us to plot the function  $\alpha(|h|)$  (Fig. 7).

#### APPENDIX D: ALGORITHM USED TO GENERATE IN-DEGREE DISTRIBUTIONS WITH SCALE-FREE TAILS

In the following, we describe the algorithm that was used to generate random networks with in-degree distributions  $p(k)$  as described in section III D. In all simulations, we applied a cutoff  $k_m = 250$ . We start with a disconnected network of size  $N$ , and repeat the following algorithm, until the network has reached an average connectivity  $\bar{K}$ :

1. Randomly select a node  $i$ . If the node has non-zero in-degree  $k_i > 0$ , disregard it and select a different

node.

2. Randomly assign a putative in-degree  $\tilde{k}_i$  with  $Prob(\tilde{k}_i) \sim \tilde{k}^{-\gamma}$ .
3. Calculate  $\bar{K}_{new} = (n_l + \tilde{k}_i)/N$ , where  $n_l$  is the number of links already assigned to the network.
4. While  $\tilde{k}_i > k_m$  or  $\bar{K}_{new} > \bar{K}$ , draw new values for  $\tilde{k}_i$ . If  $\tilde{k}_i \leq k_m$  and  $\bar{K}_{new} < \bar{K}$ , set  $k_i = \tilde{k}_i$  and assign inputs from  $k_i$  randomly selected nodes.
5. Stop if  $\bar{K}_{new} = \bar{K}$ , otherwise repeat from step 1.