Solvable Spin Model on Dynamical Networks

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We consider an Ising model in which spins are dynamically coupled by links in a network. In this model there are two dynamical quantities which arrange towards a minimum energy state in the canonical framework: the spins, s_i, and the adjacency matrix elements, c_{ij}. The model becomes exactly solvable without recourse to the replica hypothesis or other assumptions because microcanonical partition functions reduce to products of binomial factors as a direct consequence of the c_{ij}s minimizing energy. We solve the system for finite sizes and for the two possible thermodynamic limits and discuss the phase diagrams. The model can be seen as a model for social systems in which agents are not only characterized by their states but also have the freedom to choose their interaction partners in order to maximize their utility.

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Properties of many statistical systems are not solely characterized by the states of their constituents, but also depend crucially on how these interact with each other, i.e. their network (linking) structure. The way networks function can often not be fully understood by their linking structure alone because function may depend heavily on the internal states of individual nodes. It is therefore tempting to study the co-evolution of network structure and internal states. In the simplest case, this can be done in the framework of the Ising model, which immediately reminds of spin-glass models, such as the SK-model [1] or random-bond models, see e.g. [2]. Ising models where both, spins and interactions, are governed by dynamical rules have been studied assuming different timescales of evolution, where typically interaction topology 'slowly' adapts in a pre-determined way on 'fast' relaxing spins [3]. Recently, such systems have been analyzed with the replica approach in the grand-canonical ensemble assuming that the interaction topology also minimizes the energy of the system [4]; the coupling of both, spins and interactions to heat-baths at different temperatures can be treated in the respective formalism as well [5]. Note that these doubly-dynamical models are in marked contrast to the Ising model on *fixed* network structures, see [6].

Complementarily the formation of network structure driven by various Hamiltonians has been investigated in some detail [7]. We think that a full understanding of many processes taking place in networks can only be achieved in a combined approach. In the following we show that a spin system in the canonical ensemble where both, linking structure (given by the adjacency matrix c_{ij}) and spins s_i, minimize the energy, can be exactly solved without replica techniques since partition functions reduce to products of binomials.

We study the Hamiltonian

$$H(c_{ij}, s_i) = -J \sum_{i>j} c_{ij} s_i s_j - h \sum_i s_i \quad , \tag{1}$$

where sums are taken over all N nodes of the system. The position of the links in the adjacency matrix $c_{ij} \epsilon \{0, 1\}$ is a dynamical variable. The system has thus two degrees of freedom both minimizing energy: the orientation of the individual spins $s_i \epsilon \{-1, 1\}$ as usual, and the linking of spins, c_{ij}. $c_{ij} = 1(0)$ means nodes i and j are (un)connected. We consider undirected networks ($c_{ij} = c_{ji}$), the case of directed networks is a trivial extension as pointed out below. We note that a similar Hamiltonian with an additional contribution due to the grand-canonical ensemble being studied within the replica-technique has been analyzed in [4]. We denote the number of spins pointing upward by $n_\uparrow = \sum_i \theta(s_i)$, the number of links by $L = \sum_{i<j} c_{ij}$, magnetization $m = \frac{1}{N} \sum_i s_i = \frac{2n_\uparrow - N}{N}$, connectivity $c = \frac{L}{N}$, and connectedness $\varphi = \frac{L}{N^2}$. In the grand-canonical ensemble this Hamiltonian was studied in [4], however with results deviating in part substantially from the results presented here. In this work, we limit our interest to the canonical framework.

We start our analysis with the microcanonical partition function for energy E

$$\begin{aligned} \Omega(E, N, L, h) &= \sum_{\{c_{ij}\}} \sum_{\{s_i\}} \delta(H(c_{ij}, s_i) - E) \\ &= \sum_{n_\uparrow = 0}^{N} \Omega(N, n_\uparrow) \sum_{\{c_{ij}\}} \delta(H(c_{ij}, n_\uparrow) - E) \\ &= \sum_{n_\uparrow = 0}^{N} \Omega(N, n_\uparrow) \Omega(E, N, L, h, n_\uparrow) \quad , \end{aligned} \tag{2}$$

where $\Omega(N, n_\uparrow)$ is the number of configurations for a given $n_\uparrow$. $\Omega(E, N, L, h, n_\uparrow)$ denotes the microcanonical partition function for a fixed $n_\uparrow$.

In Eq.(2) it is seen that the calculation becomes greatly simplified when realizing that a fixed number of spins

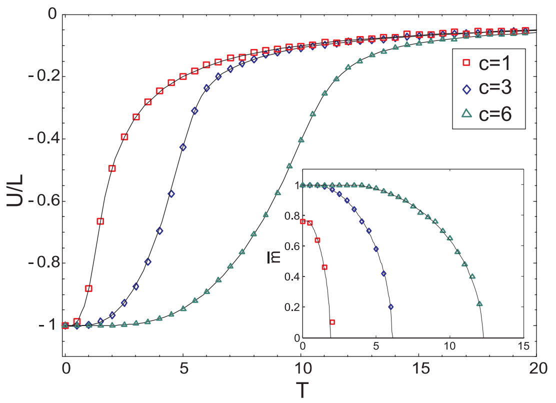

FIG. 1: Internal energy for $N = 100$, and connectivities $c = 1, 3, 6$. Solid lines correspond to the exact finite size solution, Eq. (9). Symbols are the results from a Monte Carlo simulation of the canonical ensemble pertaining to the Hamiltonian of Eq. (1). Inset: maximum of magnetization, $\bar{m}$ as a function of T (lines exact, symbols MC).

pointing upwards, $n_\uparrow$, alone is sufficient to determine the spin-state of the system since one deals with all the different topologies for a given value of $n_\uparrow$. In other words, the crucial observation is that the exact spin-configuration $\{s_i\}$ loses its relevance because the topology of the network is not fixed. In this case partition functions simply reduce to binomial factors,

$$\Omega(N, n_\uparrow) = \binom{N}{n_\uparrow} \quad , \quad \sum_{n_\uparrow=0}^{N} \binom{N}{n_\uparrow} = 2^N \quad , \tag{3}$$

and the remaining task is to determine $\Omega(E, N, L, n_\uparrow)$. To find the number of microstates leading to energy E for fixed $n_\uparrow$, the only relevant physical fact is whether a link ℓ connects two spins of (un)equal orientation, thus contributing a unit $-J$ (J) to total energy. The possible energy states are $E\epsilon\{-LJ - Nhm, -LJ + 2J - Nhm, ..., LJ - 2J - Nhm, LJ - Nhm\}$ where the lowest energy $-LJ - Nhm$ is realized if *all* links connect spins of equal orientation. In general, if k links connect spins of equal orientation ($L - k$ links connect spins of different orientation), $E = LJ - 2kJ - Nhm$. It is easy to see that the number of possible 'positions' of linking spins of equal orientation, a_e, and unequal orientation, a_u, is given by

$$\begin{aligned} a_e(N, n_\uparrow) &= \frac{1}{2}(n_\uparrow(n_\uparrow - 1) + (N - n_\uparrow)(N - n_\uparrow - 1)) \\ a_u(N, n_\uparrow) &= n_\uparrow(N - n_\uparrow) \quad , \end{aligned} \tag{4}$$

for undirected networks. Directed networks trivially follow from $a_e^{\mathrm{dir}}(N, n_\uparrow) = 2a_e(N, n_\uparrow)$ and $a_u^{\mathrm{dir}}(N, n_\uparrow) = 2a_u(N, n_\uparrow)$, because while in the undirected case, $0 < L < N(N-1)/2$, in the directed case we have, $0 < L^{\mathrm{dir}} < N(N-1)$. Each link positioned in $a_{e(u)}(N, n_\uparrow)$ contributes $-J$ (J) to the total energy E. Given Eq. (4), the microcanonical partition function for given $n_\uparrow$ and the total partition function read

$$\Omega(E, N, L, h, n_\uparrow) \binom{a_e(N, n_\uparrow)}{\frac{LJ-E-Nhm}{2J}} \binom{a_u(N, n_\uparrow)}{\frac{LJ+E+Nhm}{2J}} \quad , \tag{5}$$

$$\Omega(E, N, L, h) = \sum_{n_\uparrow=0}^{N} \binom{N}{n_\uparrow} \binom{a_e(N, n_\uparrow)}{\frac{LJ-E-Nhm}{2J}} \binom{a_u(N, n_\uparrow)}{\frac{LJ+E+Nhm}{2J}}. \tag{6}$$

We can now directly approach the problem of calculating the canonical partition function $Z(T, N, L, n_\uparrow)$ of a system with fixed $n_\uparrow$ via the Laplace transform,

$$Z(\beta, N, L) = \sum_E \sum_{n_\uparrow} \binom{N}{n_\uparrow} \Omega(E, N, L, h, n_\uparrow) e^{-\beta E} \quad . \tag{7}$$

Performing the energy summation the exact solution is

$$\begin{aligned} Z(\beta, N, L, h, n_\uparrow) = e^{LJ\beta + Nhm\beta} & \frac{\Gamma(1 + a_e)}{\Gamma(1 + L)\Gamma(1 + a_e - L)} \\ & \times {}_2\Phi_1(-a_u, -L, 1 + a_e - L, e^{-2J\beta}) \quad , \end{aligned} \tag{8}$$

with ${}_2\Phi_1(-a, b, -c, x) = \sum_{k=0}^{a} \frac{(-a)_k (b)_k x^k}{(-c)_k k!}$ the hypergeometric function and the Gamma function $\Gamma(x)$. The total canonical partition function finally is

$$Z(\beta, N, L, h) = \sum_{n_\uparrow=0}^{N} \binom{N}{n_\uparrow} Z(\beta, N, L, h, n_\uparrow) \quad , \tag{9}$$

and all thermodynamic quantities of interest are given exactly for finite sized systems, of (fixed) dimensions L and N. In Fig. 1 we show the internal energy U and magnetization as a function of temperature for different values of connectivity c as calculated from Eq. (9). Perfect agreement with Monte Carlo simulations of finite sized systems is found, where rewiring and spin-flipping have been implemented by the Metropolis algorithm. We note that for low connectivities the obtained solutions are in very good agreement with the result of independent spins, i.e. $U = c \tanh(\beta)$, as expected.

With Stirling's approximation $\binom{a}{b} \sim \left[(b/a)^{b/a}(1 - b/a)^{1-b/a}\right]^{-a}$, and the notation $y = \frac{E}{LJ}$, Eq. (7) reads

$$Z = 2^{N-2} NL(2\varphi)^{-L} \int_{-1}^{1} dm \int_{-1}^{1} dy \left[I(m, y, c, \varphi)\right]^L \quad , \tag{10}$$

with

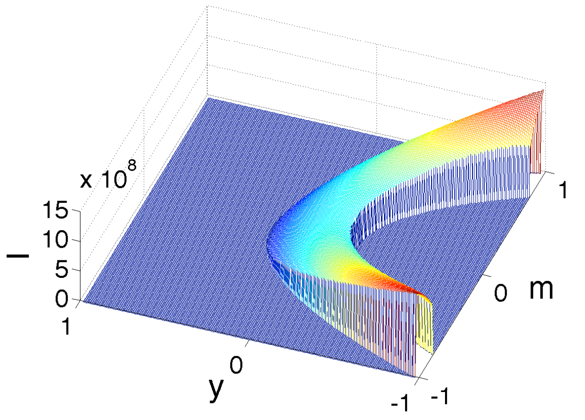

FIG. 2: Logarithm of Eq. (11) in the $m-y$ plane for $\varphi = 0.4$ and $c = 20000$. The forbidden zones are clearly visible. The maximum is always reached in the allowed zone.

$$I(m,y,c,\varphi) = \exp(-\beta J y)\left(1-m^2\right)^{-\frac{1}{2c}}\left(\frac{1-m}{1+m}\right)^{\frac{m}{2c}} \times \left(\frac{1-m^4}{1-y^2}\right)^{\frac{1}{2}}\left(\frac{(1-y)(1-m^2)}{(1+y)(1+m^2)}\right)^{\frac{y}{2}} \times \left(1-2\varphi\frac{1-y}{1+m^2}\right)^{-\frac{1}{4\varphi}(1+m^2-2\varphi(1-y))} \times \left(1-2\varphi\frac{1+y}{1-m^2}\right)^{-\frac{1}{4\varphi}(1-m^2-2\varphi(1+y))} \quad (11)$$

(h is set to zero for simplicity). In the thermodynamic limit Z is reasonably approximated by the maximal configuration, i.e. the solution to $dI/dy = 0$, which is

$$y_{\max} = \frac{-1-m^2t+\sqrt{(1+m^2t)^2-8\varphi(m^2-(2\varphi-1)t)t}}{4\varphi t} \quad (12)$$

where $t \equiv \tanh(\beta J)$. The other solution is outside the allowed parameter region of y. To ensure a real valued partition function the conditions, $1 > 2\varphi\frac{1-y}{1+m^2}$, and, $1 > 2\varphi\frac{1+y}{1-m^2}$, have to hold. Regions where they do not hold are forbidden zones in the $y-m$ plane, where the integrand of Eq. (10) is not defined, see Fig. 2. It can be shown that the maximum condition line, $y_{\max}(m)$, always stays in the allowed zone, $\forall\, -1 < m < 1$, $0 < t < 1$, and $0 < \varphi < 1/2$. It is natural to consider two distinct thermodynamical limits, $N \to \infty$, one, by keeping connectivity $c = L/N$, the other by keeping the connectedness $\varphi = L/N^2$, fixed.

$c =$ const. limit

We fix c and take $N \to \infty$. Consequently , φ vanishes, consequently Eq. (10) becomes

$$\lim_{N\to\infty} I(m,y,c,\varphi)e^{(1-\beta J y)}\left(1-m^2\right)^{-\frac{1}{2c}} \times \left(\frac{1-m}{1+m}\right)^{\frac{m}{2c}}\left(\frac{1-m^4}{1-y^2}\right)^{\frac{1}{2}}\left(\frac{(1-y)(1-m^2)}{(1+y)(1+m^2)}\right)^{\frac{y}{2}} \quad (13)$$

and the maximum condition from Eq. (12) reduces to

$$\lim_{\varphi\to 0} y_{\max} = -\frac{t+m^2}{1+m^2t} \quad . \quad (14)$$

The limiting cases for infinite and zero temperature can be worked out immediately.

The low temperature case, $\beta \gg 1$

The maximum condition further simplifies to, $y_{\max} = -1$, which when put into Eq. (13), yields

$$I\frac{1}{2}e^{1+\beta J}\left(1-m^2\right)^{-\frac{1}{2c}}\left(\frac{1-m}{1+m}\right)^{\frac{m}{2c}}\left(1+m^2\right) \quad . \quad (15)$$

To find the maxima of magnetization, we set $dI/dm = 0$, and get the solution for $T = 0$

$$\frac{1-m}{1+m} = \exp\left(-\frac{4cm}{1+m^2}\right) \quad . \quad (16)$$

The self-consistent solution is shown in Fig. 3(a): We find zero-magnetization below a critical connectivity $c < 1/2$, as well as a region where $m < 1$. One can show [8] that Eq. (16) can be obtained from the results pertaining to the grand-canonical ensemble studied in [4] in the limit of an infinitely large chemical potential.

Infinite temperature, $\beta \ll 1$

The maximum condition here becomes, $y_{\max} = -m^2$. Proceeding as before from $dI/dm = 0$, we get $0 = \frac{1}{2c}\log\left(\frac{1-m}{1+m}\right) + 2\beta J m$, which implies $m = 0$ as one solution. The other solution for small β, $m = \frac{1-2c\beta J}{2c\beta J}$, is obviously out of the allowed region for m. This means no magnetization at $T \to \infty$, $\forall c$.

The phase transition line separating the phases $m = 0$ and $m \neq 0$ is found by differentiating Eq. (13) w.r.t. m, and setting it to zero, i.e.

$$0 = \frac{1}{2c}\log\frac{1-m}{1+m} - 2m\frac{y+m^2}{1-m^4} + \frac{m(t^2-1)}{(1+m^2t)^2} \times \left[\log\frac{1-y}{1+y} + \log\frac{1-m^2}{1+m^2} - 2\beta J\right] \quad , \quad (17)$$

which reduces in the $m \to 0$ limit to the critical line given by

$$c = \frac{1}{2}\tanh^{-1}\left(\frac{J}{T_{\rm crit}}\right) \quad , \tag{18}$$

where we used $\lim_{m\to 0} y_{\rm max} = \lim_{m\to 0} -\frac{m^2+t}{1+m^2 t} = -t$. The phase diagram is shown in Fig. 3(b).

Note that in the c =const. case the free energy per node diverges logarithmically, $F/N \sim \log(N)$. However, this remains a meaningful thermodynamic limit since overextensive contributions do not affect the maximum configuration.

φ =const. limit

Fixing φ, the entropy of the system is extensive for $N \to \infty$. Fixed φ means diverging c for $N \to \infty$, and Eq. (11) becomes

$$\begin{aligned} I &= \exp(-\beta J y)\left(\frac{1-m^4}{1-y^2}\right)^{\frac{1}{2}}\left(\frac{(1-y)(1-m^2)}{(1+y)(1+m^2)}\right)^{\frac{y}{2}} \\ &\left(1-2\varphi\frac{1-y}{1+m^2}\right)^{-\frac{1}{4\varphi}(1+m^2-2\varphi(1-y))} \\ &\left(1-2\varphi\frac{1+y}{1-m^2}\right)^{-\frac{1}{4\varphi}(1-m^2-2\varphi(1+y))} \quad . \end{aligned} \tag{19}$$

Substituting the high-temperature maximum condition, $y_{\rm max} = -m^2$, one gets $(1-2\varphi)^{1-1/2\varphi} =$ const., which implies that all maximum configurations are equally probable, or, each magnetization state is equally probable at $T \to \infty$.

For finite T we hypothesize that magnetization is always extreme, $m = \pm 1$. To prove this claim we have to show that Eq. (11) is monotonically in(de)creasing in m, for $m > (<) 0$. Differentiation of (the log of) Eq. (11) w.r.t. y yields

$$\begin{aligned} &\frac{d\log(I)}{dy} - \frac{m}{2\varphi}\log\frac{1-2\varphi\frac{1-y}{1+m^2}}{1-2\varphi\frac{1+y}{1-m^2}} \\ &- \frac{1}{2}\left[\log\frac{(1+y)(1+m^2)-2\varphi(1-y^2)}{(1-y)(1-m^2)-2\varphi(1-y^2)} + 2\beta J\right] y' \quad , \end{aligned} \tag{20}$$

with

$$y' = \frac{m}{2\varphi}\left(\frac{1+m^2-4\varphi}{\sqrt{(1+m^2 t)^2 - 8\varphi(m^2-(2\varphi-1)t)t}} - 1\right) \quad . \tag{21}$$

A lengthy but trivial calculation shows that Eq. (20) is larger than zero for $m > 0$, $\forall\ 0 < \varphi < 1/2$, $0 < t < 1$, and $0 < m < 1$, where we used the fact that $y + m^2 \leq 0$. By symmetry, Eq. (20) is negative for $-1 < m < 0$. This means there is no phase transition in the thermodynamic limit for φ = const. and the system is always in a state of maximum magnetization, $m = \pm 1$.

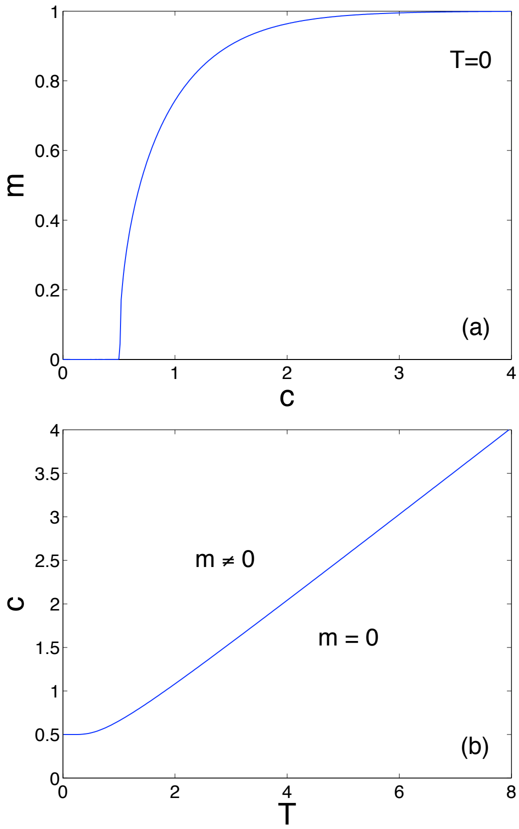

FIG. 3: Fixed-c thermodynamic limit. (a) Magnetization m as a function of connectivity c at zero temperature for $J = 1$. Below $c = 1/2$ there is no possibility for magnetization in the system. At $c \sim 4$, practically full magnetization is reached. (b) Phase diagram in the $T - c$ plane for $J = 1$.

The crucial observation of this paper is that the summation over all topologies in the Ising model on dynamical networks is equivalent to re-writing the partition function as a sum over all magnetizations. The model – which can be seen as a toy model for a variety of socio-economical situations – thus drastically reduces in complexity and becomes solvable, both for finite size and the two possible thermodynamic limits.

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