Optimal Leverage from Non-ergodicity

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In modern portfolio theory, the balancing of expected returns on investments against uncertainties in those returns is aided by the use of utility functions. The Kelly criterion offers another approach, rooted in information theory, that always implies logarithmic utility. The two approaches seem incompatible, too loosely or too tightly constraining investors' risk preferences, from their respective perspectives. This incompatibility goes away by noticing that the model used in both approaches, geometric Brownian motion, is a non-ergodic process, in the sense that ensemble-average returns differ from time-average returns in a single realization. The classic papers on portfolio theory use ensemble-average returns. The Kelly-result is obtained by considering time-average returns. The averages differ by a logarithm. In portfolio theory this logarithm can be implemented as a logarithmic utility function. It is important to distinguish between effects of non-ergodicity and genuine utility constraints. For instance, ensemble-average returns depend linearly on leverage. This measure can thus incentivize investors to maximize leverage, which is detrimental to time-average returns and overall market stability. A better understanding of the significance of time-irreversibility and non-ergodicity and the resulting bounds on leverage may help policy makers in reshaping financial risk controls.

Keywords: Portfolio selection, efficient frontier, leverage, log-optimality, Kelly criterion.

This largely expository study focuses on the simple set-up of self-financing investments, that is, investments whose gains and losses are reinvested without consumption or deposits of fresh funds, in assets undergoing geometric Brownian motion. A clear exposition of the consequences of time irreversibility in studies of risk is attempted. Understanding these consequences appears particularly important in the light of the current financial and economic crisis.

In Sec. 1 the portfolio selection problem, as introduced by Markowitz in 1952, is reviewed. Its use of utility to express risk preference is contrasted with a different ansatz, proposed by Kelly (1956) that makes use solely of the role of time. While in the terminology of modern portfolio theory, the latter ansatz can be interpreted as the assumption of logarithmic utility, in Sec. 2 the logarithm is shown to arise from the passage of time before any choices are made about risk preference. In this sense it is not the reflection of a particular investor's risk preferences but a generic null-hypothesis. Considerations of personal risk preferences can improve upon this hypothesis but they must not obscure the crucial role of time. In Sec. 3 the growth-optimal leverage that specifies a self-financing portfolio along the efficient frontier is derived and related to a minimum investment time-horizon. Finally, in Sec. 4 implications of the results from Sec. 3 for real investments are discussed and the concept of statistical market efficiency is introduced.

1. INTRODUCTION

Modern portfolio theory deals with the allocation of funds among investment assets. We assume zero transaction costs and portfolios whose stochastic properties are fully characterized by two parameters, which is equivalent to assuming that the price $p(t)$ of a portfolio is well described by geometric Brownian motion[1],

$$dp(t) = p(t)\left(\mu dt + \sigma dW_t\right), \tag{1}$$

where μ is a drift term, σ is the volatility, and

$$W(t) \equiv \int_0^t dW_t \tag{2}$$

is a Wiener process.

A portfolio i is said to be efficient [Markowitz(1952)] if
a) there exists no other portfolio j in the market with equal or smaller volatility, $\sigma_j \leq \sigma_i$, whose drift term μ_j exceeds that of portfolio i,

$$\forall j | \sigma_j \leq \sigma_i, \mu_j \leq \mu_i. \tag{3}$$

b) there exists no other portfolio j in the market with equal or greater drift term, $\mu_j \geq \mu_i$, whose volatility σ_j is smaller than that of portfolio i,

$$\forall j | \mu_j \geq \mu_i, \sigma_j \geq \sigma_i. \tag{4}$$

Markowitz (1952) argued that it is unwise to invest in any portfolio that is not efficient. In the presence of a riskless asset (with σ_i=0) all efficient portfolios lie along a straight line in the space of volatility and drift terms, that intersects the properties of the riskless asset and the properties of the so-called market portfolio [Tobin(1958)] (see Fig. 1).

Since any point along this straight line represents an efficient portfolio, Markowitz' (1952) arguments need to be augmented with additional information in order to select the optimal portfolio from the efficient frontier. This additional information is generally considered a property of the investor, namely his risk preference, represented by a utility function, $u = u\left(p(t)\right)$, that specifies the usefulness or desirability of a particular investment outcome to a particular investor[2].

In a parallel development, Kelly (1956) also considered portfolios that were described by two parameters. In his case, the portfolios were double-or-nothing

[1]Some authors define the parameters of geometric Brownian motion differently [Timmermann(1993)], precisely to circumvent the issues about to be discussed. The parameters in their notation must be carefully translated for comparisons.

[2]The concept of assigning a utility to a payoff from uncertain investments can be traced back to Bernoulli's St. Petersburg Paradox [Bernoulli(1738 (1956))]. The paradox captures the essence of the problem treated here: is an investment with infinite expected pay-off worth an infinite risk?

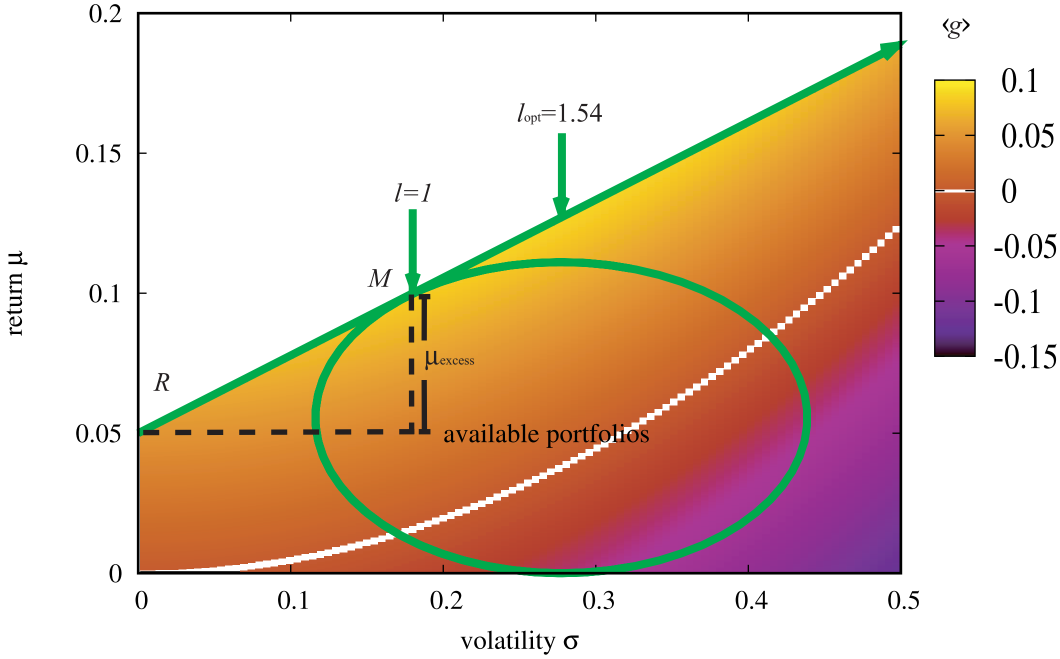

FIGURE 1.— The efficient frontier (green straight line) intersects the properties of the riskless asset R ($\mu_{\text{riskless}} = 0.05$) and is tangent to the space of available portfolios (green oval), touching at the point defined as the market portfolio, M ($\mu_{\text{excess}} = 0.05$ per investment round, $\sigma_M = 0.18$ per square-root of investment round, left arrow), corresponding to leverage $l = 1$. The color coding shows the expected growth rate, $\langle g \rangle = \mu - \sigma^2/2$; the portfolio of optimal leverage (see Sec. 3, here $l_{\text{opt}} \approx 1.54$) along the efficient frontier is indicated by the right arrow. Both for fixed volatility σ and fixed expected return μ there are no obtainable portfolios (those below the efficient frontier) whose growth rates exceed that at the efficient frontier. Zero expected growth rate is indicated white.

games on which one could bet an arbitrary fraction of one's wealth (one parameter) and knew the outcome with some probability (second parameter). Both Markowitz (1952) and Kelly (1956) recognized that it is unwise to maximize the expected return

$$\langle r \rangle = \left\langle \frac{1}{p(t)} \times \frac{dp(t)}{dt} \right\rangle, \tag{5}$$

where $\langle\rangle$ denotes the ensemble mean over realizations of the Wiener process. Markowitz rejected such strategies because the maximum-return portfolio is likely to be under-diversified [Markowitz(1952)]. In Kelly's case the probability for bankruptcy approaches one as maximum-return games are repeated [Kelly Jr.(1956)].

While Markowitz emphasized parameters such as risk preference and personal circumstances ("The proper choice among portfolios depends on the willingness and ability of the investor to assume risk." [Markowitz(1991)]), Kelly used a fundamentally different ansatz by maximizing the expected growth rate,

$$\langle g \rangle = \left\langle \frac{d \ln(p)}{dt} \right\rangle, \tag{6}$$

rather than the expected return, without an *a priori* need for additional information. The conditions under which this ansatz alone yields meaningful results have been discussed in the literature at some length and with strong opinions [Markowitz(1976), Merton and Samuelson(1974), Markowitz(1991)]. For self-financing portfolios (the focus of this study), where eventual outcomes are the product over intermediate returns these conditions are met. This is a good approximation, *e.g.* for large pension funds where fluctuations in assets under management are dominated by market fluctuations [Schwarzkopf and Farmer(2008)] and, arguably, for entire economies. Some stock market indices, for example the DAX, also reflect the value of a hypothetical constant rebalanced self-financing portfolio with zero transaction costs. Equation (6) shows that in Markowitz' framework, maximizing the growth rate corresponds to logarithmic utility $u(p(t)) = \ln(p(t))$. A more precise nomenclature would identify the logarithm as the effect of time-averaging, and subsequently impose utility on this quantity.

2. TWO AVERAGES

For riskless assets, the chain rule of ordinary calculus implies that (Eq. 5) and (Eq. 6) are identical. The (expected) growth rate of a riskless asset is thus equal to its (expected) rate of return, $\langle g_{\text{riskless}} \rangle = \langle r_{\text{riskless}} \rangle$.

Interpreting the expression for the expected return in (Eq. 5) is not entirely straight-forward since $W(t)$ of (Eq. 2) is non-differentiable (to physicists "dW_t/dt" is known as Gaussian white noise). To simplify notation, we use the convention that an ensemble average, $\langle\rangle$, operating on dW_t is interpreted as acting on $W(t)$ first (before differentiation),

$$\left\langle \frac{dW_t}{dt} \right\rangle = \frac{d\left\langle W(t) \right\rangle}{dt} = 0, \tag{7}$$

which expresses the zero-drift property of $W(t)$, although the object $\frac{dW_t}{dt}$ does not exist, strictly speaking. This is not an unusual convention; in the physics literature, for instance, it is common to write the equivalent of $\langle dW_t/dt\rangle = 0$, see [van Kampen(1992)], Ch. 9, and [Parisi(1988)], Ch. 19, especially note 3, for a more detailed discussion.

Combining (Eq. 1) and (Eq. 5), we now compute the expectation value of the fractional price increment per infinitesimal time step, the expected return,

$$\begin{aligned} \langle r \rangle &= \left\langle \frac{1}{p(t)} \times \frac{p(t)\mu dt + p(t)\sigma dW_t}{dt} \right\rangle \\ &= \mu + \sigma \left\langle \frac{dW_t}{dt} \right\rangle \\ &= \mu. \end{aligned} \tag{8}$$

The stochastic case has to be treated even more carefully because the chain rule of ordinary calculus does not apply. The object $d(\ln(p))$ in (Eq. 6) has to be evaluated using Itô's lemma[3]. With the chain rule of ordinary calculus replaced by Itô's version, (Eq. 5) and (Eq. 6) now correspond to different averages.

Itô's lemma for Eq. 1, which we need to evaluate (Eq. 6), takes the form[4]

$$df = \left(\frac{\partial f}{\partial t} + \mu p \frac{\partial f}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 f}{(\partial p)^2} \right) dt + p\sigma \frac{\partial f}{\partial p} dW_t, \tag{9}$$

where $f = f(p(t), t)$ is some function of the Itô process $p(t)$ of Eq. 1, and time t. The dependencies of $p(t)$ and $f(p(t), t))$ have been left out in Eq. 9 to avoid clutter. The third term in (Eq. 9) constitutes the difference from the increment $d\tilde{f}$ for a function $\tilde{f}(\tilde{p}, t)$ of a deterministic process $\tilde{p}$. Due to the second derivative, Itô's lemma takes effect if and only if $f(p(t), t)$ is non-linear in p. To derive the increment $d(\ln(p))$, we need to choose $f(p(t), t) \equiv \ln(p(t))$, that is, a non-linear function. We arrive at

$$d(\ln(p(t))) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t. \tag{10}$$

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[3]Since the following arguments depend on the use of Itô's lemma, we stress that Itô's interpretation of increments like Eq. 1 is indeed the appropriate choice in the present context because it implies statistical independence of $p(t)$ and the increment dW_t. Alternative interpretations are possible, notably Stratonovich's, but they define different dynamics. For a detailed discussion, see [van Kampen(1992)], Ch. 9, [Lau and Lubensky(2007)] and [Øksendal(2005)], Ch. 3 and Ch. 5.

[4]This calculation can be found in any textbook on financial derivatives, *e.g.* [Hull(2006)] Ch. 12.

The corresponding average,

$$\begin{aligned}\langle g\rangle &= \left\langle \frac{d(\ln(p(t)))}{dt} \right\rangle \\ &= \mu - \frac{\sigma^2}{2} + \sigma \left\langle \frac{dW_t}{dt} \right\rangle \\ &= \mu - \frac{\sigma^2}{2},\end{aligned} \tag{11}$$

is called the expected growth rate, or geometric mean return in the literature[5]. Equation (11) shows that while the expected return enters into the growth rate, it does so in combination with the volatility. Equation 11 thus quantifies for the present set-up the statement that large returns and small volatilities are desirable. This is illustrated in Fig. 1. We observe that in the deterministic case, the two averages coincide – with $\sigma = 0$, (Eq. 11) is equal to Eq. (Eq. 8), as expected from the chain rule of ordinary calculus.

2.1. *Discussion of the two averages*

How can we make sense of the difference between the expected return computed in (Eq. 8) and the expected growth rate of (Eq. 11) in the presence of non-zero volatility?

The expected return, μ, computed in (Eq. 8) is an ensemble average over all realizations of the stochastic increments dW_t, disregarding the time-irreversibility of the process. Stochasticity, in this average, is integrated out before it takes effect through Itô's lemma. Equation (8) is the answer to the following question: "what is the return of this investment, averaged over all possible universes?", where a universe is defined as a particular sequence of events, *i.e.* one realization of the process (Eq. 2). Since we only live in one realization of the universe but stay alive in that universe for some time, for most of us this question is less relevant than the question: "what is the return of this investment, averaged over time?", to which the answer is Eq. 11. This is illustrated in Fig. 2, where one realization of a self-financed portfolio is compared, over 350 investment rounds, to an average over an increasing number of universes. This is achieved by producing independent sequences of wealth, corresponding to resetting an investor's wealth and starting over again. The independent sequences are then averaged (arithmetically) at equal times. As this averaging procedure over universes destroys stochasticity, the stochastic exponential growth process (whose expected growth rate was calculated using Eq. 11) approaches deterministic exponential growth (whose growth rate is the expected return, Eq. 8). The procedure of starting over

[5] More generally, the expected growth rate of a process whose returns obey any probability distribution is the geometric mean of that distribution, see *e.g.* [Kelly Jr.(1956), Markowitz(1976)]. The concepts presented here are thus not restricted to the case of portfolios characterized by two parameters.

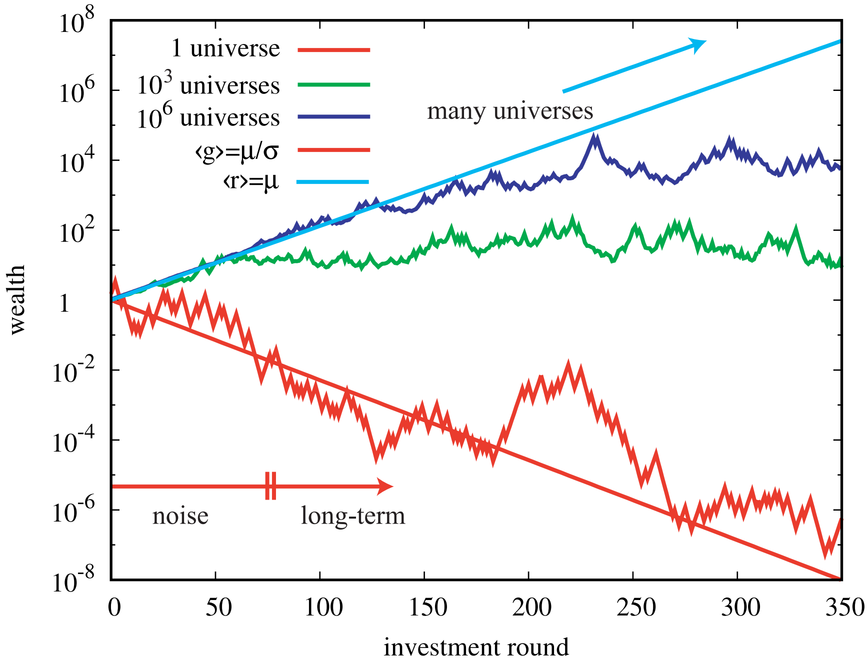

FIGURE 2.— Wealth starts at unity at time 0 and then behaves according to the following rules: In each time step, or investment round, 40% of the wealth accumulated up to time t is lost with probability 1/2, otherwise 50% is gained. This corresponds to $\mu = 0.05$ per investment round, volatility $\sigma = 0.45$ per square root of investment round, and expected exponential growth rate $\langle g \rangle \approx -0.051$. For short times the performance is noise-dominated; after about 75 rounds (red arrow, (Eq. 13)) the expected growth rate takes over (see also Fig. Fig. 3). To uncover the expected return from the dynamics, an average over many independent universes must be taken.

again is like going back in time, or periodically resetting one's investment to its initial value. But the self-financing portfolios considered here do not allow such resetting[6].

The expected growth rate of Eq. 11, on the other hand, can be interpreted as a long-time average in a single realization of the process. The stochastic equation of

[6]There are situations in which the ensemble-average is more relevant. Kelly construed the example of a gambler whose wife, once a week, gives him an allowance of one dollar to bet on horses [Kelly Jr.(1956)]. The optimal strategy for this gambler is to maximize the expected return, (Eq. 8). The reason is that the gambler resets his wealth in each round of the game instead of reinvesting. His wealth is the sum (a linear object) of past gains, whereas under re-investment it would be the product (an object that is non-linear and hence affected by Itô's lemma). In Sec. 3 we will see how this translates into preferred values of leverage.

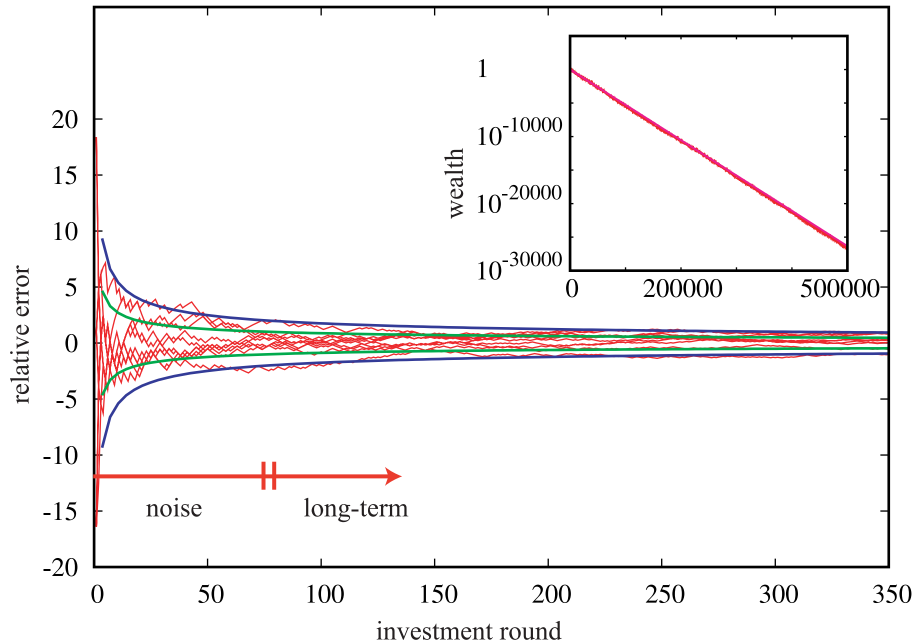

FIGURE 3.— Relative errors in estimates $\frac{g(t)-\langle g\rangle}{\langle g\rangle}$ of the growth rates, using the path-independent relation $g(t)=\frac{1}{t}\ln\left(\frac{p(t)}{p(0)}\right)$ in several realizations of the process described in the caption of Fig. 2. Green lines show one standard deviation from expected estimates, $\frac{1}{\sqrt{t}}\times\frac{\sigma}{\mu-\frac{\sigma^2}{2}}$, blue lines show two standard deviations. Inset: Long-time averages approach deterministic behavior with the expected growth rate.

motion, Eq. 1, is solved by integrating Eq. 10 over time and then exponentiating,

$$p(t)=p(0)\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right). \tag{12}$$

The Wiener process $W(t)$ of Eq. 2 is Gaussian-distributed with mean 0 and standard deviation $\sqrt{t}$. In the exponent of the solution, Eq. 12, time thus enters linearly via the deterministic term $\left(\mu-\frac{\sigma^2}{2}\right)t$ but only as the square root via the stochastic term $\sigma W(t)$. Therefore, in time random deviations from the expected growth rate will diminish. This is illustrated in Fig. Fig. 3.

At zero volatility the "expected" (certain) growth rate equals the return. This is probably the origin of the erroneous intuition that expected return should be maximized. At zero volatility, any investor will choose the highest-yielding portfolio available, maximizing both the growth rate and the expected return over all possible universes (there is only one possible universe now). Generalizing to the stochastic case, it is still sensible to maximize growth rates, but this is not equivalent to maximizing expected returns. A good guide, whether investing with or without volatility, is concern for the future, rather than concern for copies of oneself in parallel universes.

In the literature, logarithmic utility is often associated with the term "long-term investment". The long term here means a time scale that is long enough for the deterministic part of the exponent in (Eq. 12) to dominate over the noise. Applying this terminology to the present case, one is investing either for the long term, or in a regime where randomness dominates – the latter case, which is not typically given a name, may be described as "gambling". If in (Eq. 12), the $W(t)$ is replaced with $\sqrt{\langle W(t)^2 \rangle} = \sqrt{t}$, we estimate that gambling stops, and the long term begins when

$$t > t_c = \frac{\sigma^2}{\left(\mu - \frac{\sigma^2}{2}\right)^2}. \tag{13}$$

In Fig. 2 the corresponding time scale is $t_c \approx 75$ investment rounds, indicated by the break in the red arrow. At this point, the typical relative error in estimates of $\langle g \rangle$ based on past performance is unity, indicated by a break in the arrow in Fig. 3. In a single universe, such as our reality, the system is never dominated by the expected return – neither in the short run nor in the long run. Instead, there is an initial noise-regime where no significant trends can be discerned, where-after the expected growth rate dominates the performance.

3. Time scales and optimal leverage from non-ergodicity

The use of the null model of maximizing the growth rate eliminates the *a priori* need to specify risk-preferences. Tailoring real-life investments to real investors' needs does require difficult to formalize knowledge of their circumstances, but a number of issues can be illuminated without such knowledge, in the simple context of the null-hypothesis. For instance, a well-defined optimal leverage can be computed as will be shown now. The result follows directly from Kelly's (1956) arguments, and several authors have come to the same conclusions using different methods *e.g.* [Thorp(2006), Kestner(2003)]. In addition, the characteristic time scale of (Eq. 13) is calculated for the leveraged case, which defines a critical leverage where the expected growth rate vanishes.

Any efficient portfolio along the straight efficient frontier can be specified by its fractional holdings of the market portfolio [Sharpe(1964)], which we define as the leverage, l. For instance, an investor who keeps all his money in the riskless asset holds a portfolio of leverage $l = 0$; half the money in the riskless asset and half in the market portfolio is leverage $l = 0.5$, and borrowing as much money as one owns and investing everything in the market portfolio corresponds to $l = 2$ etc.

The expected return in the leveraged case can be written as the sum $\mu_{\text{riskless}} + l\mu_{\text{excess}}$, where μ_{excess}, is the excess rate of return of the market portfolio over the riskless rate of return. At zero leverage, only the riskless return enters; the excess return is added in proportion to the leverage. Noting that both the expected return and the volatility depend linearly on the leverage, we obtain the leveraged stochastic process

$$dp_l(t) = p_l(t)\left(\mu_{\text{riskless}} + l\mu_{\text{excess}})dt + l\sigma_M dW_t\right), \tag{14}$$

where σ_M is the volatility of the market portfolio. Just like with Eq. 1, we can use Itô's lemma, (Eq. 9), to derive the equation of motion for the logarithm of the price, p_l, of the leveraged portfolio,

$$d(\ln(p_l)) = \left(\mu_{\text{riskless}} + l\mu_{\text{excess}} - \frac{l^2\sigma_M^2}{2}\right)dt + l\sigma_M dW_t. \tag{15}$$

The expected leveraged exponential growth rate is thus[7]

$$\begin{aligned} \langle g_l \rangle &\equiv \left\langle \frac{d(\ln(p_l))}{dt} \right\rangle \\ &= \left(\mu_{\text{riskless}} + l\mu_{\text{excess}} - \frac{l^2\sigma_M^2}{2}\right). \end{aligned} \tag{16}$$

The positive contribution to $\langle g_l \rangle$ is linear in the leverage, but the negative contribution is quadratic in the leverage. The quadratic term enters through Itô's lemma and makes the expected growth rate non-monotonic in the leverage.

Markowitz (1952) rejected maximum-expected-return strategies because the corresponding portfolios are likely to be under-diversified and hence to have an unacceptably high volatility. Equation 16 shows that in the limit of large leverage, seeking high expected returns, the growth rate along the efficient frontier diverges negatively, as $\lim_{l\to\infty} \langle g_l \rangle / l^2 = -\sigma_M^2/2 < 0$.

To find the optimal leverage, we differentiate Eq. 16 with respect to l and set the result to zero, obtaining

$$l_{\text{opt}} = \mu_{\text{excess}}/\sigma_M^2. \tag{17}$$

The second derivative of Eq. 16 with respect to l is $-\sigma_M^2$, which is always negative, implying that $\langle g_l \rangle$ corresponding to l_{opt} is maximized. This calculation shows that there exists a privileged portfolio along the efficient frontier. If the market portfolio has a lower volatility than the portfolio of maximum growth rate, as in Fig. 1, then the wise investor will leverage his position by borrowing ($l_{\text{opt}} > 1$). If, on the other hand, the market portfolio has a higher volatility, as in Fig. 2, then he will keep some fraction of his money safe ($l_{\text{opt}} < 1$).

[7]This can also be seen immediately by replacing in Eq. 11, $\mu \to \mu_{\text{riskless}} + l\mu_{\text{excess}}$, $\sigma \to l\sigma_M$

We note that $\langle g \rangle$ can never be increased by decreasing returns at constant volatility, nor can it be increased by increasing volatility at constant return, because from Eq. 11, $\partial_\mu \langle g \rangle > 0$ and $\partial_\sigma \langle g \rangle < 0$. Therefore, the globally (*i.e.* selected from all possible portfolios) growth-optimal portfolio will always be located on the efficient frontier. This optimization thus does not contradict but does refine modern portfolio theory for the generic case of self-financing portfolios.

Including the leverage in (Eq. 13) results in the leveraged characteristic time scale separating the gambling from the growth-rate dominated regime

$$t_c^l = \frac{l^2 \sigma_M^2}{\left(\mu_{\text{riskless}} + l\mu_{\text{excess}} - \frac{l^2\sigma_M^2}{2}\right)^2}. \tag{18}$$

This time scale diverges at the critical leverages,

$$l_c^{\pm} = \frac{\mu_{\text{excess}} \pm \sqrt{\mu_{\text{excess}}^2 + 2\mu_{\text{riskless}}\sigma_M^2}}{\sigma_M^2} \tag{19}$$

We are interested only in the positive branch, l_c^+, here[8]. At this leverage, the expected leveraged growth rate (the denominator of (Eq. 18)) is zero, wherefore the system will forever be dominated by noise. For larger leverages, $l > l_c^+$, the expected growth rate is negative. This is what happens in Fig. 2, where the optimum $l_{\text{opt}} \approx 0.25$, the critical leverage is $l_c \approx 0.49$, and the system runs at $l = 1$.

4. DISCUSSION

Practically relevant lessons from the above considerations may be learnt from the extremes. A 100% mortgage, for instance, corresponds to infinite leverage on the borrower's investment (assuming that the purchase is not part of a larger investment portfolio). Although total loss on a home-purchase only means dipping into negative equity, certain financial products that have become popular in recent years must be regarded as irresponsible.

Real portfolios of constant leverage (apart from $l = 0$ and possibly $l = 1$) need to be constantly rebalanced as the value of the market portfolio fluctuates and changes the fraction of wealth invested in it. Holding any such portfolio is costly, both in terms of monitoring time and in terms of transaction costs. For applications, the above considerations thus need to be adapted, even if real prices are well described by (Eq. 1). The value of real optimal leverage depends on the investor's ability to balance portfolios, which is affected by the available technology and, due to market impact, by the volume of the investment. The assumptions made in this study are likely to lead to an over-estimate of optimal leverage: Equation (17) was derived in continuous time, corresponding to truly

[8]The negative branch corresponds to zero expected growth rate in a negatively leveraged portfolio, consisting of the riskless asset and a small short position in the market portfolio.

constantly rebalanced portfolios, zero transaction costs were assumed, log-normal return-distributions, certain knowledge of μ and σ, and no risk premiums charged on money borrowed for leveraging.

Using some broad index like the S&P500 or the DJIA as a proxy for the market portfolio, a typical excess return over US government bonds is $\mu_{\text{excess}} \approx 0.05$ p.a., while the volatility may be $\sigma \approx 0.18 \sqrt{\text{p.a.}}$ (like in Fig. 1 with investment rounds of one year duration). This implies an optimal leverage, as calculated above, of 1.54, but it is unlikely that the simple strategy of borrowing money and investing it in the S&P500 would outperform the market. A reasonable guess is that real optimal leverage is closer to 1. It will now be argued that $l_{\text{opt}} = 1$ is a possible attractor for a self-organized market system. How could such statistical market efficiency work? If the market's optimal leverage is greater than unity, money will be borrowed to be invested. This situation can arise as a consequence of low interest rates and low-cost credit. Borrowed money tends to increases volatility due to additional trading to maintain leverage, and potential margin calls and similar constraints on investors [Geanakoplos(1997)]. Thus, as investors increase their leverage, they reduce optimal leverage, creating a negative feedback loop whose strength depends on the magnitude of the impact of leverage on volatility ("$\frac{d\sigma}{dl}$"), this brings optimal leverage down, $l_{\text{opt}} \to 1^+$. Conversely, if optimal leverage is less than unity, investors will sell risky assets, thereby reducing prices and increasing expected returns, such that optimal leverage increases, possibly up to $l_{\text{opt}} \to 1^-$.

The current financial crisis has been related to a continued extension of credit [Soros(2008b), Soros(2008a)], *i.e.* increasingly leveraged investments. In some areas leverage rose far beyond optimality, creating an unstable market situation. The crisis may thus be viewed as a response where volatility has suddenly increased, reducing optimal leverage, which in turn has led to deleveraging and falling prices.

The use of leverage is not fundamentally constrained by the prevailing framework of portfolio selection because it relies on a necessarily and explicitly subjective notion of optimality, dependent on utility. This has become problematic because excessive leveraging has been encouraged by asymmetric reward structures. Securitization, for example, creates such structures by separating those who sell leverage products (such as mortgages) from those who eventually bear the risk associated with them. Since the sellers are rewarded for every sale, irrespective of the risk, securitization is conducive to over-leveraged market situations. Similarly, an investment manager who benefits from gains in the portfolio managed by him but is not personally liable for losses has an incentive to exceed growth-optimal leverage (see footnote on p. 7). In the ideal set-up discussed above, the growth-optimal ansatz suggests a simple reward scheme through alignment of interests: requiring investment managers to invest all their wealth in the accounts managed by them. It is thereby achieved that the growth-optimal investment strategy for the account is also growth optimal for the investment manager.

Equation (13) gives an order-of magnitude estimate for the beginning of the long term where the trend of the market becomes distinguishable from its fluctuations. Under the parameters of Fig. 1 this characteristic time scale t_c is somewhere between 4 and 5 years. Comparisons between portfolios with similar stochastic properties are meaningful only on much longer time scales.

If the investment horizon is shorter than the characteristic time scale t_c for some product, then the investment will essentially be a gamble, wherefore t_c may be viewed as a minimum investment horizon. At the critical leverage l_c, see (Eq. 19), this minimum investment horizon becomes infinite, meaning that any such investment is a gamble. For $l > l_c$ the time scale is finite and marks the transition between noise and discernible loss of invested capital.

In conclusion, the use of utility functions should be preceded by considerations of time-irreversibility. The concept of many universes is a useful tool to understand the limited significance of ensemble-average returns on risky investments. While common rhetoric seems to hold that excessive leverage arises when investors are short-term oriented, there is no benefit from leveraging beyond optimality even in the short term – this regime is noise-dominated, not expected-return dominated. Thus, it would be harmless, in principle, to reward investment managers daily, hourly or indeed continuously for their performance – as long as they share in the risks as much as in the rewards. A privileged portfolio exists somewhere along the efficient frontier, characterized by an optimal leverage and a maximized expected growth rate, the advantages of which have also been discussed elsewhere [Breiman(1961), Merton and Samuelson(1974), Cover and Thomas(1991)]. While modern portfolio theory does not preclude the use of what it calls logarithmic utility, it seems to underemphasize its fundamental significance. It was pointed out here that the default choice to optimize the growth rate need not be interpreted as genuine logarithmic utility, but is physically motivated by the passage of time.

5. ACKNOWLEDGEMENTS

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