The Mixed Strategy Equilibria and Adaptive Dynamics in the Bar Problem

John Q. Cheng

SFI WORKING PAPER: 1996-08-063

The mixed strategy equilibria and adaptive dynamics in the Bar Problem*
(extended draft)

John Cheng
Department of Economics
University of Michigan
Ann Arbor, MI 48109
jcheng@econ.lsa.umich.edu

January 24, 1997

Abstract

This paper looks at an N-person coordination game which is called by Brian Arthur as the bar problem. We look at the mixed strategy equilibria and show that there is a unique purely mixed strategy equilibrium in which all players play non-degenerated mixed strategies. We also examine some simple dynamics that might evolve the system to that equilibrium by conducting some preliminary numerical experiments. The results show that under usual initial conditions the system will converge to the unique mixed strategy equilibrium.

*This paper summarizes the project I did during the 1995 Santa Fe Institute Computational Economics Workshop. Special thanks to the organizers of the Workshop, John Miller and Scott Page. I also wish to thank fellow workshop participants, especially Nick Feltovich and Jordan Rapporport, for their very helpful comments and suggestions.

1 Introduction

Brian Arthur[?] used a very simple yet interesting problem to illustrate effective uses of inductive reasoning as well as computational methods in economics research. Our interest here is to examine Arthur's example (called by Arthur as the Bar Problem) from a different angle.

There is a bar called El Farol in downtown Santa Fe. There are n agents interested in going to the bar each night. All agents have identical preferences. Each of them will enjoy the night at El Farol very much if there are no more than m agents in the bar; however, each of them will suffer miserably if there are more than m agents. In Arthur's example, the total number of agents is 100, and the threshold number $m = 60$. The only information available to agents is the number of visitors to the bar in previous nights. The payoffs would be 1 for each agent who made the right choice at any particular night (i.e., go to the bar when the total number of agents going is less than or equal to m, or staying at home if the total attendance turns out to be more than m); for those made the wrong choice, the payoff would be 0. We call this the **BP game**.

Arthur investigated the number of agents attending the bar over time by using a diverse population of simple rule based agents. One interesting result he obtained is that over time, the average attendance of the bar is about 60.[1] Arthur examined the dynamic driving force behind this "equilibrium", but he did not give a precise mathematical explanation in his paper.

Our interest here is the mixed-strategy equilibria in this coordination game, and the possible dynamic processes that could lead to such equilibria. I will characterize some preliminary results in the following sections. I shall start with some theoretical analysis and discuss some computer experiments we have conducted.[2]

2 The BP game and its equilibria

The bar problem is in the essence a N-agent coordination game in which each agent has two possible pure strategies.

We are all familiar with the 2×2 coordination game with symmetric payoffs.[3] There are two pure strategy Nash equilibria and one mixed strategy Nash equilibrium.

As the number of players go up in this symmetric pure coordination game, the number of pure strategy equilibria goes up quickly. In the Bar Problem, the number of pure strategy Nash equilibria is C_n^m, which is the possible number of ways to pick m out of n agents.

2.1 Strategic complementarity

The BP game is in fact a game with strategic complementarity, using the definition in Milgrom and Shannon[?]. It is straightforward to check that this game with its payoff structure satisfied the so called "single crossing property" and some other related conditions, and therefore has the desired strategic complementarity. (For the definition of "single crossing property", see [?].)

In Milgrom and Roberts[?], it is shown as a theorem that in any adaptive and sophisticated learning process (as defined in their paper) associated with a normal form game, the strategies of players will eventually stay in a region[4] defined by the set of pure strategy Nash equilibria of that underlying normal form game, provided that the game is one with strategic complementarity.

[1] Arthur did point out in his paper that this could be explained as each agents playing a mixed strategy.

[2] All results presented in this paper were obtained while attending the Santa Fe workshop.

[3] We restrict attention to games with symmetric payoffs in this paper.

[4] The size of the region is the key.

While this provides powerful insights and results for learning and dynamic processes in some games, it doesn't tell us much about the BP game. Because there are so many pure strategy Nash equilibria in the BP game, the target region (see Milgrom and Roberts[**?**] for the precise definition) defined by the pure strategy Nash equilibria is in fact the whole strategy space. Hence the theorem mentioned above by Milgrom and Roberts wouldn't help too much in the BP game.

2.2 Mixed strategy equilibria

We discuss the number of mixed strategy Nash equilibria in the BP game.

Define

$$F(m,n,\alpha) \equiv \sum_{0}^{m-1} C_{n-1}^{i}\alpha^{i}(1-\alpha)^{n-1-i}$$

It is easy to verify that[5]

$$\begin{aligned} F(m,n,0) &= 1 \\ F(m,n,1) &= 0 \end{aligned}$$

Proposition 1

$$\frac{\partial F}{\partial \alpha} < 0$$

Proof. 1

$$\begin{aligned} F_{\alpha} &= \frac{\partial}{\partial \alpha}(\sum_{i=0}^{m-1} C_{n-1}^{i}\alpha^{i}(1-\alpha)^{n-1-i}) \\ &= \sum_{i=1}^{m-1} i \cdot C_{n-1}^{i}\alpha^{i-1}(1-\alpha)^{n-1-i} - \sum_{i=0}^{m-1}(n-1-i)\cdot C_{n-1}^{i}\alpha^{i}(1-\alpha)^{n-2-i} \\ &= \sum_{i=1}^{m-1}(n-1)\cdot C_{n-2}^{i-1}\alpha^{i-1}(1-\alpha)^{n-1-i} - \sum_{i=0}^{m-1}(n-1)\cdot C_{n-2}^{i}\alpha^{i}(1-\alpha)^{n-2-i} \\ &= (n-1)[\sum_{j=0}^{m-2} C_{n-2}^{j}\alpha^{j}(1-\alpha)^{n-2-j} - \sum_{i=0}^{m-1} C_{n-2}^{i}\alpha^{i}(1-\alpha)^{n-2-i}],\ here\ j = i-1. \\ &= (n-1)\cdot(-1)C_{n-2}^{m-1}\alpha^{m-1}(1-\alpha)^{n-m-1} \\ &< 0 \end{aligned}$$

If in a mixed strategy equilibrium, all players are choosing the same "mixing", then we call it a symmetric mixed strategy equilibrium. As to the number of mixed strategy equilibria in the BP game, we have the following result.

Proposition 2 *The number of symmetric mixed strategy equilibrium in the bar problem is one.*

[5]strictly speaking, this should be $\lim_{\alpha\to 0} F(m,n,\alpha) = 1$.

Proof. 2 *Let us consider the situation with a total number of agents being n and the threshold number being m, assuming $n > m$.*

In a symmetric equilibrium, all agents choose the same probability of going to the bar, which we denote as α.

In such a situation, we must have the following condition which says that each agent is indifferent between going and staying.

$$1 \cdot \sum_{i=0}^{m-1} C_{n-1}^{i} \alpha^{i} (1-\alpha)^{n-1-i} = 1 \cdot \sum_{i=m}^{n-1} C_{n-1}^{i} \alpha^{i} (1-\alpha)^{n-1-i}$$

The left hand side is the payoff of an agent if it chooses to go, the right hand side is its payoff if it chooses to stay.

Notice that the sum of both sides is exactly one. Therefore we know,

$$\sum_{i=0}^{m-1} C_{n-1}^{i} \alpha^{i} (1-\alpha)^{n-1-i} = \frac{1}{2}$$

or

$$F(m,n,\alpha) = \frac{1}{2}$$

Since $F(m,n,\alpha)$ is a continuous function in α, and $F_\alpha < 0$, $F(m,n,\alpha) = \frac{1}{2}$ has a unique solution. Hence the number of symmetric mixed strategy equilibrium is one.

The next result goes a little further.

Proposition 3 *In a mixed strategy equilibrium of the BP game, if no agent plays pure strategy, then all agents must play the same mixed strategy.*

Proof. 3 *We take advantage of the fact that there are only two pure strategies available to each agent.*

If there are two agents using different strategies in a mixed strategy equilibrium, we can denote them as agent 1 and 2, and their probability of going to the bar as α_1 and α_2. It must be that $\alpha_1 \neq \alpha_2$.

Let us use T to denote the group of all players other than players 1 and 2.

Since player 1 is using a mixed strategy, it must be that the probability of more than m-1 players show up, given T and α_2, equals $\frac{1}{2}$. Since it has to be true that both events of more than $m-1$ agents show up and no more than $m-1$ agents show up are equally likely, given the strategies of T and player 2.

Likewise, player 2 is also using a mixed strategy. It follows that the probability of more than m-1 players show up, given T and α_1, equals $\frac{1}{2}$. Since it has to be true that both events of more than $m-1$ agents show up and no more than $m-1$ agents show up are equally likely, given the strategies of T and player 1.

Since all agents decide independently in a mixed strategy equilibrium, and $\alpha_1 \neq \alpha_2$, the conclusion of the above two paragraphs cannot both be true. For example, if $\alpha_1 > \alpha_2$, and the probability of more than m-1 players show up equals $\frac{1}{2}$ (given T and α_2), then it is impossible to have the probability of more than m-1 players show up, given T and α_1, also equals $\frac{1}{2}$. Notice that agents only have two available pure strategies.

This contradiction tells us that in an equilibrium where all agents play mixed strategies, they must be playing the same mixed strategy.

Combining the previous two propositions, we know that there is a *unique* equilibrium in the BP game in which all players play (the same) non-degenerated mixed strategy.

3 Agents with mixed strategy

Since we know that the BP game has only one mixed strategy equilibrium (MNE) and many pure strategy equilibria (PNE), it is natural to conjecture that the unique MNE is a focal point of the game.

In particular, we are interested to see if any population of agents playing mixed strategies facing selection pressure will evolve to the state in which all agents play the MNE strategy. The reasons that we are interested in agents using mixed strategy are summarized as the following:

1. No agent can commit to any single pure strategy, especially when dealing with a large population. (More difficult to achieve coordination as the number of agents increase.)

2. Agents could appear to be using mixed strategies.

3. Reasonable agents might believe their own analysis only to some degree, thus might some doubts about their own prediction. Mixed strategy captures this degree of doubt, with pure strategies as the limiting beliefs.

4. The use of behavioral mixed strategy, at least to some people, seems reasonable when agents have to make decisions under uncertainty.

5. The use of mixed strategy in the BP game seems no more unreasonable than the use of, say, the use of myopic optimization in evolutionary games.

We used some computational experiments to investigate whether agents playing mixed strategies will end up in the MNE.

4 Computational experiments

4.1 The set up

We[6] conducted some numerical experiments with 100 agents, each of them are restrict to select a mixed strategy from the interval of $[0, 1]$. The threshold number is set to be 60.

Agents are randomly assigned a number from $[0, 1]$ as their initial strategy, which is just the probability of going to the bar. Let's denote this number as α_i for agent i. These are the initial conditions of our experiments.

Then, in each round, agents decide whether they want to go to the bar by drawing a number randomly from $[0, 1]$, and compare this number with their α's. If α_i is larger than the number agent i drew, it would decide to go to the bar; if not, it would choose to stay.

In each round, each agent would receive a score of 1 if it made the right choice and 0 if it made the wrong choice. Whether their choices are right or wrong depends on the threshold value, other agents' choices, and their own choices.

Each generation of agents can live for 10 periods.[7] At the end of each generation, all agents are ranked by their accumulative score.

[6] I am grateful to David Kane, Micheal Herron, and other fellow participants in the Santa Fe workshop for helping me with important computational issues.

[7] 10 is an arbitrary choice and can be changed to other numbers.

The selection procedure we used in our experiments is very simple: take the top 50 percent[8] of agents and eliminate the rest. We duplicate each of the surviving agent to get a new *group* of 100 agents.

The next step is to allow mutation in agent's strategy. It is done by first choosing a random number from the interval $[-b, +b]$, where b is randomly chosen from 0 to 0.01. Then add this number to the number that representing the current strategy of an agent, with the restriction that all strategies have to remain in the interval $[0, 1]$. Now we have a new *generation* of 100 agents.[9]

In each of our experiment, we ran this set up for a large number of generations to obtain the final outcome.

4.2 The results

The preliminary experiments we ran involved either 1000 or $100,000$ generations.

The outcomes for the experiments with $100,000$ generations indicate that the final strategies of agents are within some tolerance of the MNE strategy, regardless of the initial condition of the experiments (i.e., the distribution of the initial strategies of the agents).

The outcomes for the experiments with 1000 generations indicate some sign of convergence. Instead of collapsing into a region around the MNE strategy, the final strategies of agents often (in most cases) have two "accumulation" points.

5 Conclusion and further research directions

Systematic experiments are yet to be conducted to gain more insights and results about the transition to the mixed strategy equilibrium.

The results of the preliminary experiments indicates that the unique mixed strategy Nash equilibrium is an attraction point for agents playing the BP game. The uniqueness plays an important role here.

Some of the experimental set-up could be modified, and suggestions have been made to incorporate random mutation steps and more variations of the initial conditions.

There are many important issues this paper chose not to explore, some of them could generate further interesting results. These issues include:

- Incorporate the possibility that agents' actions are correlated by some public signal.
- Incorporate more systematic learning into agents' strategy.
- Have a population of heterogeneous agents.
- Consider more complex stage games. In particular, consider games where agents have more than 2 possible pure strategies and/or games with asymmetric payoffs.

[8]we also experimented with other non-trivial (neither 0 nor 1) percentages, and the result are qualitatively the same.

[9]There could be many other ways to apply a selection pressure on the agent population, but our experiments conducted were restricted in using only the method mentioned above.