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**q -GENERALIZATION OF SYMMETRIC α -STABLE DISTRIBUTIONS.
PART I**

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Abstract

The classic and the Lévy-Gnedenko central limit theorems play a key role in theory of probabilities, and also in Boltzmann-Gibbs (BG) statistical mechanics. They both concern the paradigmatic case of probabilistic independence of the random variables that are being summed. A generalization of the BG theory, usually referred to as nonextensive statistical mechanics and characterized by the index q ($q = 1$ recovers the BG theory), introduces global correlations between the random variables, and recovers independence for $q = 1$. The classic central limit theorem was recently q -generalized by some of us. In the present paper we q -generalize the Lévy-Gnedenko central limit theorem.

1 Introduction

In the recent paper by some of us [1], a generalization of the classic central limit theorem applicable to nonextensive statistical mechanics [2, 3] (which recovers the usual, Boltzmann-Gibbs statistical mechanics as the $q = 1$ particular instance), was presented. We follow here along the lines of that paper. One of the important aspects of this generalization is that it concerns the case of globally correlated random variables. On the basis of the q -Fourier transform F_q introduced there (F_1 being the standard Fourier transform), and the function

$$z(s) = \frac{1+s}{3-s},$$

we described attractors of conveniently scaled limits of sums of q -correlated random variables¹ with a finite $(2q-1)$ -variance². This description was essentially based on the mapping

$$F_q : \mathcal{G}_q[2] \rightarrow \mathcal{G}_{z(q)}[2], \quad (1)$$

¹ q -correlation corresponds to standard probabilistic independence if $q = 1$, and to specific global correlations if $q \neq 1$.

²We required there $q < 2$. Denoting $Q = 2q - 1$, it is easy to see that this condition is equivalent to the finiteness of the Q -variance with $Q < 3$.

where $\mathcal{G}_q[2]$ is the set of q -Gaussians (the number 2 in the notation will soon become transparent).

In the current paper, which consists of two parts, we will introduce and study a q -analog of the α -stable Lévy distributions, and establish a q -generalization of the Lévy-Gnedenko central limit theorem. In this sense, the present paper is a conceptual continuation of paper [1]. The classic theory of the α -stable distributions was originated and developed by Lévy, Gnedenko, Feller and others (see, for instance, [4, 5, 6, 7, 8] and references therein for details and history). The α -stable distributions found a huge number of applications in various practical studies [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], confirming the frequent nature of these distributions.

For simplicity we will analyse only symmetric densities in the one-dimensional case. Stable distributions with skewness and multivariate stable distributions can be studied in the same way applying the known classic techniques.

In Part 1 we study a q -generalization of the α -stable Lévy distributions. Namely, we consider the symmetric densities $f(x)$ with asymptotics $f \sim C|x|^{-\frac{1+\alpha}{1+\alpha(q-1)}}$, $|x| \rightarrow \infty$, where C is a positive constant³. We classify these distributions in terms of their densities depending on the parameters $q < 2$ (or equivalently $Q < 3$, $Q = 2q - 1$) and $0 < \alpha \leq 2$. We establish the mapping

$$F_q : \mathcal{G}_{q^L}[2] \rightarrow \mathcal{G}_q[\alpha], \quad (2)$$

where $\mathcal{G}_q[\alpha]$ is the set of all densities $\{be_q^{-\beta|\xi|^\alpha}, b > 0, \beta > 0\}$, and

$$q^L = \frac{3 + Q\alpha}{1 + \alpha}, \quad Q = 2q - 1,$$

i.e.,

$$\frac{2}{q^L - 1} = \frac{1 + \alpha}{1 + \alpha(q - 1)}.$$

The particular case $q = Q = 1$ recovers $q^L = \frac{3+\alpha}{1+\alpha}$, already known in the literature [3]. We consider the values of parameters Q and α ranging in the set

$$\mathcal{Q}_0 = \{(Q, \alpha) : -1 < Q < 3, 0 < \alpha < 2, \alpha < \frac{2}{1 - Q}\}.$$

The values of Q and α in

$$\mathcal{Q}_2 = \{(Q, \alpha) : -1 < Q < 3, \alpha = 2\}$$

were studied in [1]. Note that for Q and α in

$$\mathcal{Q}_1 = \{(Q, \alpha) : \frac{2}{1 - Q} \leq \alpha < 2, -1 < Q < 0\},$$

the densities have finite $(2q - 1)$ -variance. Consequently, the theorem obtained in [1] is again applicable. For $(Q, \alpha) \in \mathcal{Q}_0$ the Q -variance is infinite. We will focus our analysis namely on this case. Note that the case $\alpha = 2$, in the framework of the present description like in that of the classic α -stable distributions, becomes peculiar.

In Part 2 we study the attractors of scaled sums, and expand the results of the paper [1] to the region

$$\mathcal{Q} = \{(Q, \alpha) : -1 < Q < 3, 0 < \alpha \leq 2\},$$

generalizing the mapping (1) into the form

$$F_{\zeta_\alpha(q)} : \mathcal{G}_q[\alpha] \rightarrow \mathcal{G}_{\zeta_\alpha(q)}[\alpha], \quad q < 2, 0 < \alpha \leq 2, \quad (3)$$

³Hereafter $g(x) \sim h(x), x \rightarrow a$, means that $\lim_{x \rightarrow a} \frac{g(x)}{h(x)} = 1$.

where

$$\zeta_\alpha(s) = \frac{\alpha - 2(1 - q)}{\alpha} \text{ and } z_\alpha(s) = \frac{\alpha q + 1 - q}{\alpha + 1 - q}.$$

Note that, if $\alpha = 2$, then $\zeta_2(q) = q$ and $z_2(q) = (1 + q)/(3 - q)$, thus recovering the mapping (1), and consequently, the result of the paper [1].

These two types of q -generalized descriptions of the standard symmetric α -stable distributions, based on mappings (2) and (3) respectively, allow us to draw a full picture of the q -generalization of the Lévy-Gnedenko central limit theorem that we have obtained.

2 Basic operations of q -mathematics

We recall briefly basics of q -mathematics. Indeed, the analysis we conduct is entirely based on the q -structure (for more details see [20, 21, 22, 23, 24] and references therein). Let x and y be two given real numbers. By definition, the q -sum of these numbers is defined as $x \oplus_q y = x + y + (1 - q)xy$. The q -sum is commutative, associative, recovers the usual summing operation if $q = 1$ (i.e. $x \oplus_1 y = x + y$), and preserves 0 as the neutral element (i.e. $x \oplus_q 0 = x$). By inversion, we can define the q -subtraction as $x \ominus_q y = \frac{x - y}{1 + (1 - q)y}$. The q -product for x, y is defined by the binary relation $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$. This operation also commutative, associative, recovers the usual product when $q = 1$, and preserves 1 as the unity. The q -product is defined if $x^{1-q} + y^{1-q} \geq 1$. Again by inversion, it can be defined the q -division: $x \oslash_q y = (x^{1-q} - y^{1-q} + 1)^{\frac{1}{1-q}}$. Note that, for $q \neq 1$, $x \otimes_q 0 \neq 0$, and division by zero is allowed.

3 q -generalisation of the exponential and cyclic functions

Now we introduce the q -exponential and q -logarithm [20], which play an important role in the nonextensive theory. These functions are denoted by e_q^x and $\ln_q x$ and respectively defined as $e_q^x = [1 + (1 - q)x]_+^{\frac{1}{1-q}}$ and $\ln_q x = \frac{x^{1-q} - 1}{1 - q}$, ($x > 0$). Here the symbol $[x]_+$ means that $[x]_+ = x$ if $x \geq 0$, and $[x]_+ = 0$ if $x < 0$. We mention the main properties of these functions, which we will use essentially in this paper. For the q -exponential the relations $e_q^{x \oplus_q y} = e_q^x e_q^y$ and $e_q^{x+y} = e_q^x \otimes_q e_q^y$ hold true. These relations can be written equivalently as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ ⁴, and $\ln_q(xy) = (\ln_q x) \oplus_q (\ln_q y)$. The q -exponential and q -logarithm have the asymptotics

$$e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2), \quad x \rightarrow 0, \quad (4)$$

and

$$\ln_q(1 + x) = x - \frac{q}{2}x^2 + o(x^2), \quad x \rightarrow 0, \quad (5)$$

respectively. If $q < 1$, then, for real x , $|e_q^{ix}| \geq 1$ and $|e_q^{ix}| \sim (1 + x^2)^{\frac{1}{2(1-q)}}$, $x \rightarrow \infty$. Similarly, if $q > 1$, then $0 < |e_q^{ix}| \leq 1$ and $|e_q^{ix}| \rightarrow 0$ if $|x| \rightarrow \infty$.

Lemma 3.1 *Let $A_n(q) = \prod_{k=0}^n a_k(q)$ where $a_k(q) = q - k(1 - q)$. Then there holds the series expansion*

$$e_q^x = 1 + x + x^2 \sum_{n=0}^{\infty} \frac{A_n(q)}{(n+2)!} x^n, \quad \forall x \in \mathbb{R}.$$

⁴This property reflects the possible extensivity of S_q in the presence of special correlations [25, 26, 27, 28].

Corollary 3.2 For arbitrary real number x the equation

$$e_q^{ix} = \{1 - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n}\} + i\{x - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}\}$$

holds.

Define q -cos and q -sin by formulas

$$\cos_q(x) = 1 - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n}, \quad (6)$$

and

$$\sin_q(x) = x - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}. \quad (7)$$

Properties of q -sin, q -cos, and corresponding q -hyperbolic functions, were studied in [22]. Here we note that the q -analogs of the well known Euler's formulas read

Corollary 3.3 (i) $e_q^{ix} = \cos_q(x) + i \sin_q(x)$;

$$(ii) \cos_q(x) = \frac{e_q^{ix} + e_q^{-ix}}{2};$$

$$(iii) \sin_q(x) = \frac{e_q^{ix} - e_q^{-ix}}{2i}.$$

Lemma 3.4 The following equality holds:

$$\cos_q(2x) = e_{2q-1}^{2(1-q)x^2} - 2 \sin_{2q-1}^2(x). \quad (8)$$

Proof. The proof follows from the definitions of $\cos_q(x)$ and $\sin_q(x)$, and from the fact that $(e_q^x)^2 = e_{(1+q)/2}^{2x}$ (see Lemma 2.1 in [1]).

Denote $\Psi_q(x) = \cos_q 2x - 1$. It follows from Equation (8) that

$$\Psi_q(x) = (e_{2q-1}^{2(1-q)x^2} - 1) - 2 \sin_{2q-1}^2(x). \quad (9)$$

Lemma 3.5 Let $q \geq 1$. Then we have

1. $-2 \leq \Psi_q(x) \leq 0$;
2. $\Psi_q(x) = -2q x^2 + o(x^3), x \rightarrow \infty$.

Proof. It follows from (9) that $\Psi_q(x) \leq 0$. Further, $\sin_q(x)$ can be written in the form (see [22]) $\sin_q(x) = \rho_q(x) \sin[\varphi_q(x)]$, where $\rho_q(x) = (e_q^{(1-q)x^2})^{1/2}$ and $\varphi_q(x) = \frac{\arctan(1-q)x}{1-q}$. This yields $\Psi_q(x) \geq -2$ if $q \geq 1$. Using the asymptotic relation (4), we get

$$e_{2q-1}^{2(1-q)x^2} - 1 = 2(1-q)x^2 + o(x^3), x \rightarrow 0. \quad (10)$$

It follows from (7) that

$$-2 \sin_{2q-1}^2(x) = -2x^2 + o(x^3), x \rightarrow 0. \quad (11)$$

The relations (9), (10) and (11) imply the second part of the statement.

4 q -Fourier transform for symmetric functions

The q -Fourier transform, based on the q -product, was introduced in [1] and played a central role in establishing the q -analog of the standard central limit theorem. Formally the q -Fourier transform for a given function $f(x)$ is defined by the formula

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx. \quad (12)$$

For discrete functions $f_k, k = 0, \pm 1, \dots$, this definition takes the form

$$F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e_q^{ik\xi} \otimes_q f(k). \quad (13)$$

In the future we use the same notation in both cases. We also call (12) or (13) the q -characteristic function of a given random variable X with an associated density $f(x)$, using the notations $F_q(X)$ or $F_q(f)$ equivalently.

It should be noted that, if in the formal definition (12), f is compactly supported, then integration has to be taken over this support, although, in contrast with the usual analysis, the function $e_q^{ix\xi} \otimes_q f(x)$ under the integral does not vanish outside the support of f . This is an effect of the q -product.

The following lemma establishes the relation of the q -Fourier transform without using the q -product.

Lemma 4.1 *The q -Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{(f(x))^{1-q}}} dx. \quad (14)$$

Remark 4.2 *Note that, if the q -Fourier transform of a given function $f(x)$ defined by the formal definition in (12) exists, then it coincides with the expression in (14). The q -Fourier transform determined by the formula (14) has an advantage when compared to the formal definition: it does not use the q -product, which is, as we noticed above, restrictive in use. From now on we refer to (14) when we speak about the q -Fourier transform.*

Further to the properties of the q -Fourier transform established in [1], we note that, for symmetric densities, the assertion analogous to Lemma 4.1 is true with the q -cos.

Lemma 4.3 *Let $f(x)$ be a symmetric density. Then its q -Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \cos_q(x\xi[f(x)]^{q-1}) dx. \quad (15)$$

Proof. Notice that, because of the symmetry of f ,

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx = \int_{-\infty}^{\infty} e_q^{-ix\xi} \otimes_q f(x) dx.$$

Taking this into account, we have

$$F_q[f](\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \left(e_q^{ix\xi} \otimes_q f(x) + e_q^{-ix\xi} \otimes_q f(x) \right) dx.$$

Applying Lemma 4.1 we obtain

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \frac{e_q^{ix\xi[f(x)]^{q-1}} + e_q^{-ix\xi[f(x)]^{q-1}}}{2} dx,$$

which coincides with (15).

Let us now refer to the three sets:

$$\mathcal{Q}_0 = \{(Q, \alpha) : -1 < Q < 3, 0 < \alpha < 2, \alpha < \frac{2}{1-Q}\},$$

$$\mathcal{Q}_1 = \{(Q, \alpha) : \frac{2}{1-Q} \leq \alpha < 2, -1 < Q < 0\},$$

$$\mathcal{Q}_2 = \{(Q, \alpha) : -1 < Q < 3, \alpha = 2\},$$

where $Q = 2q - 1$. Obviously $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ gives the semi-strip

$$\mathcal{Q} = \{-1 < Q < 3, 0 < \alpha \leq 2\}$$

which contains the top boundary. A q -generalization of the central limit theorem for Q and α in the set \mathcal{Q}_2 was studied in [1]. It is not hard to verify that any density corresponding to $(Q, \alpha) \in \mathcal{Q}_1$ has a *finite* Q -variance. Hence, for \mathcal{Q}_1 also, the theorem obtained in [1] is applicable. For $(Q, \alpha) \in \mathcal{Q}_0$, the Q -variance of densities considered in the following lemma is *infinite*. From now on, we focus our studies on this case.

Lemma 4.4 *Let $f(x)$, $x \in \mathbb{R}$, be a symmetric probability density function of a given random vector. Further, let either*

(i) *the $(2q - 1)$ -variance $\sigma_{2q-1}^2(f) < \infty$, (associated with $\alpha = 2$), or*

(ii) *$f(x) \sim C(|x|^{-\frac{\alpha+1}{1+\alpha(q-1)}})$, $|x| \rightarrow \infty$, with $C > 0$ and $(2q - 1, \alpha) \in \mathcal{Q}_0$.*

Then, for the q -Fourier transform of $f(x)$, the following asymptotic relation holds true:

$$F_q[f](\xi) = 1 - \mu_{q,\alpha} |\xi|^\alpha + o(|\xi|^\alpha), \xi \rightarrow 0, \quad (16)$$

where

$$\mu_{q,\alpha} = \begin{cases} \frac{q}{2} \sigma_{2q-1}^2 \nu_{2q-1}, & \text{if } \alpha = 2; \\ \frac{2^{2-\alpha}(1+\alpha(q-1))}{2-q} \int_0^\infty \frac{-\Psi_q(y)}{y^{\alpha+1}} dy, & \text{if } 0 < \alpha < 2. \end{cases} \quad (17)$$

with $\nu_{2q-1}(f) = \int_{-\infty}^\infty [f(x)]^{2q-1} dx$.

Remark 4.5 *Stable distributions require $\mu_{q,\alpha}$ to be positive. We have seen (Lemma 3.5) that if $q \geq 1$, then $\Psi_q(x) \leq 0$ (not being identically zero), which yields $\mu_{q,\alpha} > 0$. If $q = 0$, we can check by straightforward calculations that $\mu_{q,\alpha} = 0$. We denote by \mathcal{Q}_0^+ the subset of \mathcal{Q}_0 , where $\mu_{q,\alpha} > 0$.*

Proof. First, assume $\alpha = 2$. Evaluate $F_q[f](\xi)$. Using Lemma 4.1 we have

$$F_q[f](\xi) = \int_{-\infty}^{\infty} (e_q^{ix\xi}) \otimes_q f(x) dx = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} dx. \quad (18)$$

Making use of the asymptotic expansion (4) we can rewrite the right hand side of (18) in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \left(1 + \frac{ix\xi}{[f(x)]^{1-q}} - q/2 \frac{x^2 \xi^2}{[f(x)]^{2(1-q)}} + o\left(\frac{x^2 \xi^2}{[f(x)]^{2(1-q)}}\right) \right) dx =$$

$$1 - (q/2) \xi^2 \sigma_{2q-1}^2 \nu_{2q-1} + o(\xi^2), \quad \xi \rightarrow 0,$$

from which the first part of Lemma follows.

Now, assume $(2q-1, \alpha) \in \mathcal{Q}_0^+$. Apply Lemma 4.3 to obtain

$$F_q[f](\xi) - 1 = \int_{-\infty}^{\infty} f(x) [\cos_q(x\xi[f(x)]^{q-1}) - 1] dx =$$

$$2 \int_0^N f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx + 2 \int_N^{\infty} f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx,$$

where N is a sufficiently large finite number. In the first integral we use the asymptotic relation $\Psi(\frac{x}{2}) = -\frac{q}{2}x^2 + o(x^3)$, which follows from Lemma 3.5, and get

$$2 \int_0^N f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx =$$

$$-q\xi^2 \int_0^N x^2 f^{2q-1}(x) dx + o(\xi^3), \quad \xi \rightarrow 0. \quad (19)$$

In the second integral taking into account the hypothesis of the lemma with respect to $f(x)$, we have

$$2 \int_N^{\infty} f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx = 2 \int_N^{\infty} \frac{1}{x^{\frac{\alpha+1}{1+\alpha(q-1)}}} \Psi_q\left(\frac{x^{1-\frac{(\alpha+1)(q-1)}{1+\alpha(q-1)}} \xi}{2}\right) dx.$$

We use the substitution

$$x^{\frac{2-q}{1+\alpha(q-1)}} = \frac{2y}{\xi}$$

in the last integral, and obtain

$$2 \int_N^{\infty} f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx =$$

$$-\frac{2^{2-\alpha}(1+\alpha(q-1))}{2-q} |\xi|^\alpha \int_0^\infty \frac{-\Psi_q(y)}{y^{\alpha+1}} dy + o(\xi^\alpha), \quad \xi \rightarrow 0. \quad (20)$$

Hence, the obtained asymptotic relations (19) and (20) complete the proof.

5 (q, α) -stable distributions

Two random variables X and Y are called to be q -correlated if

$$F_q[X + Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi). \quad (21)$$

In terms of densities, relation (21) can be rewritten as follows. Let f_X and f_Y be densities of X and Y respectively, and let f_{X+Y} be the density of $X + Y$. Then

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f_{X+Y}(x) dx = F_q[f_X](\xi) \otimes_q F_q[f_Y](\xi). \quad (22)$$

Definition 5.1 A random variable X is said to have a (q, α) -stable distribution if its q -Fourier transform is represented in the form $b e_q^{-\beta|\xi|^\alpha}$, with some real constants $b > 0$ and $\beta > 0$. We denote by $\mathcal{L}_q(\alpha)$ the set of all (q, α) -stable distributions.

Denote $\mathcal{G}_q(\alpha) = \{b e_q^{-\beta|\xi|^\alpha}, b > 0, \beta > 0\}$. In other words $X \in \mathcal{L}_q(\alpha)$ if $F_q[f] \in \mathcal{G}_q(\alpha)$. We will study limits of sums

$$Z_N = \frac{1}{D_N(q)} (X_1 + \dots + X_N), N = 1, 2, \dots$$

where $D_N(q)$, $N = 1, 2, \dots$, are some reals (scaling parameter), that belong to $\mathcal{L}_q(\alpha)$, when $N \rightarrow \infty$.

Definition 5.2 A sequence of random variables Z_N is said to be q -convergent to a (q, α) -stable distribution, if $\lim_{N \rightarrow \infty} F_q[Z_N](\xi) \in \mathcal{G}_q(\alpha)$ locally uniformly by ξ .

Theorem 1. Assume $(2q - 1, \alpha) \in \mathcal{Q}_0^+$. Let $X_1, X_2, \dots, X_N, \dots$ be symmetric random variables mutually q -correlated and all having the same probability density function $f(x)$ satisfying the conditions of Lemma 4.4.

Then Z_N , with $D_N(q) = (\mu_{q,\alpha} N)^{\frac{1}{\alpha(2-q)}}$, is q -convergent to a (q, α) -stable distribution, as $N \rightarrow \infty$.

Remark 5.3 By definition \mathcal{Q}_0 excludes the value $\alpha = 2$. The case $\alpha = 2$, in accordance with the first part of Lemma 4.4, coincides with Theorem 2 of [1]. Note in this case $\mathcal{L}_q(2) = \mathcal{G}_{q^*}(2)$, where $q^* = \frac{3q-1}{q+1}$.

Proof. Assume $(Q, \alpha) \in \mathcal{Q}_0^+$. Let f be the density associated with X_1 . First we evaluate $F_q(X_1) = F_q(f(x))$. Using Lemma 4.4 we have

$$F_q[f](\xi) = 1 - \mu_{q,\alpha} |\xi|^\alpha + o(|\xi|^\alpha), \xi \rightarrow 0. \quad (23)$$

Denote $Y_j = N^{-\frac{1}{\alpha}} X_j, j = 1, 2, \dots$. Then $Z_N = Y_1 + \dots + Y_N$. Further, it is readily seen that, for a given random variable X and real $a > 0$, there holds $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$. It follows from this relation that $F_q(Y_j) = F_q[f](\frac{\xi}{(\mu_{q,\alpha} N)^{1/\alpha}})$, $j = 1, 2, \dots$. Moreover, it follows from the q -correlation of Y_1, Y_2, \dots (which is an obvious consequence of the q -correlation of X_1, X_2, \dots) and the associativity of the q -product that

$$F_q[Z_N](\xi) = F_q[f](\mu_{q,\alpha} N^{-\frac{1}{\alpha}} \xi) \otimes_q \dots \otimes_q F_q[f](\mu_{q,\alpha} N^{-\frac{1}{\alpha}} \xi) \quad (N \text{ factors}). \quad (24)$$

Hence, making use of the expansion (5) for the q -logarithm, Eq. (24) implies

$$\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f](\mu_{q,\alpha} N^{-\frac{1}{\alpha}} \xi) = N \ln_q(1 - \frac{|\xi|^\alpha}{N} + o(\frac{|\xi|^\alpha}{N})) =$$

$$-|\xi|^\alpha + o(1), N \rightarrow \infty, \quad (25)$$

locally uniformly by ξ .

Hence, locally uniformly by ξ ,

$$\lim_{N \rightarrow \infty} F_q(Z_N) = e_q^{-|\xi|^\alpha} \in \mathcal{G}_q(\alpha). \quad (26)$$

Thus, Z_N is q -convergent to a (q, α) -stable distribution, as $N \rightarrow \infty$. Q.E.D.

This theorem links the classic Lévy distributions with their q_α^L -Gaussian counterparts. Indeed, in accordance with this theorem, a function f , for which

$$f \sim C/x^{(\alpha+1)/(1+\alpha(q-1))}, \quad |x| \rightarrow \infty,$$

is in $\mathcal{L}_q(\alpha)$, i.e. $F_q[f](\xi) \in \mathcal{G}_q(\alpha)$. It is not hard to verify that there exists a q_α^L -Gaussian, which is asymptotically equivalent to f . Let us now find q_α^L . Any q_α^L -Gaussian behaves asymptotically like $C/|x|^\eta = C/|x|^{2/(q_\alpha^L-1)}$, $C = \text{const}$, i.e. $\eta = 2/(q_\alpha^L - 1)$. Hence, we reobtain the relation

$$\frac{\alpha + 1}{1 + \alpha(q - 1)} = \frac{2}{q_\alpha^L - 1}. \quad (27)$$

Solving this equation with respect to q_α^L , we have

$$q_\alpha^L = \frac{3 + Q\alpha}{\alpha + 1}, \quad Q = 2q - 1, \quad (28)$$

linking three parameters: α , the parameter of the α -stable Lévy distributions, q , the parameter of correlation, and q_α^L , the parameter of attractors in terms of q_α^L -Gaussians (see Fig. 2). Equation (28) identifies all (Q, α) -stable distributions with the same index of attractor $G_{q_\alpha^L}$ (See Fig. 1).

In the particular case $Q = 1$, we recover the known link between the classical Lévy distributions ($q = Q = 1$) and corresponding q_α^L -Gaussians. Put $Q = 1$ in Eq. (28) to obtain

$$q_\alpha^L = \frac{3 + \alpha}{1 + \alpha}, \quad 0 < \alpha < 2. \quad (29)$$

When α increases between 0 and 2 (i.e. $0 < \alpha < 2$), q_α^L decreases between 3 and 5/3 (i.e. $5/3 < q_\alpha^L < 3$): See Figs. 2 and 4(a).

It is useful to find the relationship between $\eta = \frac{2}{q_\alpha^L - 1}$, which corresponds to the asymptotic behaviour of the attractor, and (α, Q) . Using formula (28), we obtain (Fig. 3)

$$\eta = \frac{2(\alpha + 1)}{2 + \alpha(Q - 1)}. \quad (30)$$

If $Q = 1$ (classic Lévy distributions), then $\eta = \alpha + 1$, as well known.

Analogous relationships can be obtained for other values of Q . We call, for convenience, a (Q, α) -stable distribution to be a Q -Cauchy distribution, if its parameter $\alpha = 1$. We obtain the classic Cauchy-Poisson distribution if $Q = 1$. The corresponding line can be obtained cutting the surface in Fig. 3 along the line $\alpha = 1$. For Q -Cauchy distributions we have

$$q_1^L(Q) = \frac{3 + Q}{2} \quad \text{and} \quad \eta = \frac{4}{Q + 1}, \quad (31)$$

respectively (see Figs. 2 and 3).

The relationship between α and q_α^L for typical fixed values of Q are given in Fig. 4 (a). In this figure we can also see, that $\alpha = 1$ (Cauchy) corresponds to $q_1^L = 2$ (in the $Q = 1$ curve). In Fig. 4 (b) the relationships between Q ($Q = 2q - 1$) and q_α^L are represented for typical fixed values of α .

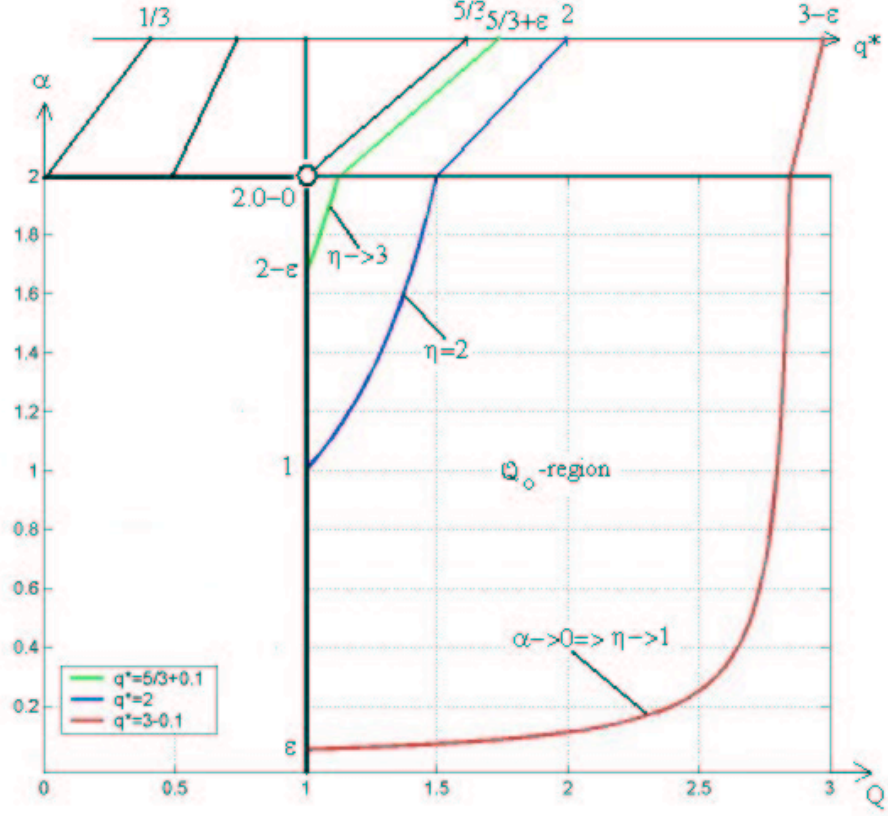


Figure 1: All pairs of (Q, α) on the indicated curves are associated with the same q_α^L -Gaussian. Two curves corresponding to two different values of q_α^L do not intersect. In this sense these curves represent the constant levels of q_α^L or $\eta = 2/(q_\alpha^L - 1)$. The line $\eta = 1$ joins the points $(Q, \alpha) = (1, 0.0 - 0)$ and $(3 - 0, 2)$; the line $\eta = 2$ joins the Cauchy distribution (noted C) with itself at $(Q, \alpha) = (1, 1)$ and at $(2, 2)$; the $\eta = 3$ line joins the points $(Q, \alpha) = (1, 2.0 - 0)$ and $(5/3, 2)$ (by ϵ we simply mean to give an indication, and not that both infinitesimals coincide). The entire line at $Q = 1$ and $0 < \alpha < 2$ is mapped into the line at $\alpha = 2$ and $5/3 \leq q^* < 3$.

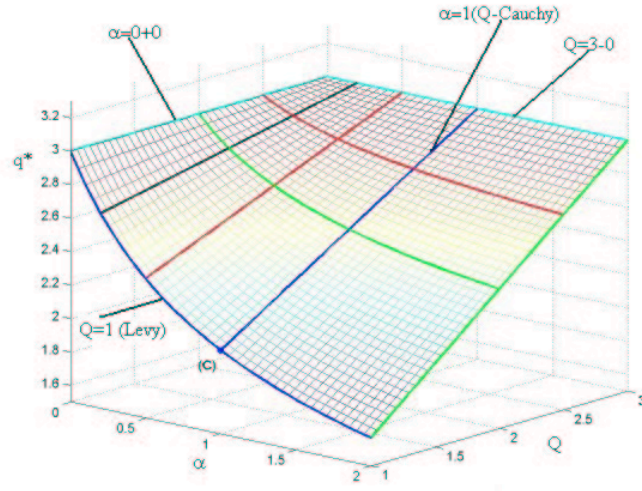


Figure 2: $q_\alpha^L \equiv q^*$ as function of (Q, α) .

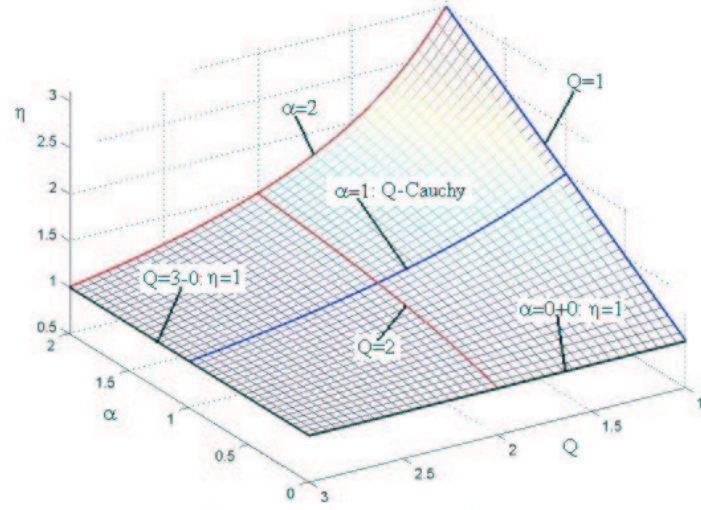


Figure 3: η as the function of (Q, α) .

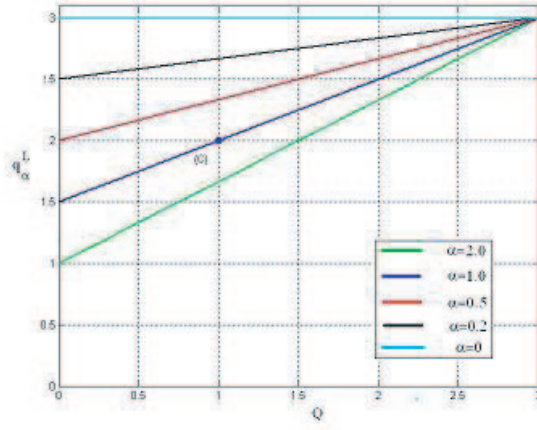
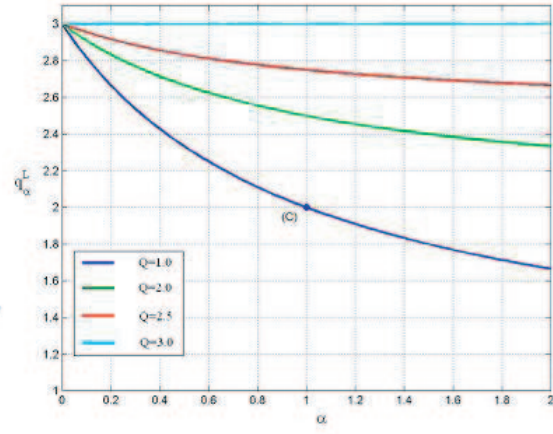


Figure 4: Constant Q and constant α sections of Fig. 2.

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