The Topology of Cities

Christa Brelsford
Taylor Martin
Joe Hand
Luís M. A. Bettencourt

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Authors: Christa Brelsford[1], Taylor Martin[2], Joe Hand[1], Luís M. A. Bettencourt[1*]

Affiliations:
[1]Santa Fe Institute, 1399 Hyde Park Rd, Santa Fe NM 87501, USA.

[2]Department of Mathematics, Sam Houston State University, Huntsville, Texas 77341, USA.

*Correspondence to: bettencourt@santafe.edu @BettencourtLuis

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Abstract:
Is there an ideal city form? As cities proliferate worldwide this has become a central question underpinning sustainable development and economic opportunity for billions of people. We provide extensive empirical evidence, mathematical analysis and a set of theorems to show that the answer to this question is topological, not geometric. We show that cities can be decomposed into two types of networked spaces – accesses and places - and prove that these spaces display universal topological characteristics common to all cities, provided specific mathematical conditions are met. While exceptions to these conditions are rare in developed cities, many urban slums fall into a different topological class. This expresses the central difficulty of developing cities as a rigorous mathematical problem that we show how to solve optimally through the introduction of infrastructure networks into city blocks at minimal disruption and cost.

One sentence Summary:
All cities are shown mathematically to share a common topology, manifested by transformations between any two street layouts and the gradual evolution of their neighborhoods.

Cities exist in many different geometries (1), from regular grids to curvy mazes of streets and alleys. City geometries are shaped by processes that are essential for all cities, such as the circulation of people, goods and information; and processes that are particular to each place, like geography, defense and the choices made by planners in response to crisis, growth, or change (1, 2). This diversity of form is aesthetically appealing (1–3) but has defied quantitative explanations or replication by known practices. A long history of urban planning (1–6) and, more recently, of data-driven statistical analysis (6–9) has attempted to classify urban spatial patterns with an eye towards optimal design, but the quest for ideal forms has remained elusive.

Similarly, the morphology of other complex systems, from river basins (10, 11) to vascular systems (12, 13), to neural networks (14, 15) or human social organizations (16) shows substantial variation around ideal templates derived from encoding minimal energy dissipation or optimal communication. Thus, we need to identify what is essential versus what is variable in the architecture of complex systems (17).

Here we show that the essential spatial architecture of cities is dictated by topology, not geometry. Topology provides a general quantitative measure of surfaces emphasizing the equivalence between diverse shapes when they can be continuously deformed into each other (18, 19). Consequently, topological invariance allows for considerable freedom of form so long as certain spatial relationships are preserved.

Graph theory and topology are intimately connected branches of mathematics (18, 19). They were born out the need for formal tools to analyze relationships between (urban) spaces. By transforming specific places into nodes and accesses between them (bridges) into edges, Leonhard Euler created a graph representing Königsberg in 1735, which he used to prove the non-existence of a walk crossing each of its seven bridges only once (19). This showed for the first time how questions about the relationships between physical spaces -topological problems- can be analyzed and answered regardless of geometric details through the analysis of graphs.

Taking this as our starting point, we derive here the general topology of cities. We divide urban built space into two categories (20): i) *access systems* (roads and paths), and ii) *places* (buildings, public spaces). These two spaces span the entire city and are interconnected (Fig. S1-3) as each other's negative spaces. Then, we can conceive each city as a series of interconnected blocks, each of which is an "island" surrounded by infrastructure that mediates access to each place internal to the block, Fig. 1. We now show how the topology of cities can be systematically characterized by successively considering the topology of their access systems, followed by the relationship between places and their accesses and finally among different places (parcels) within each block.

The topology of the access system is relatively simple. The physical volume of all paths and roads is a connected 2D surface: any point on its surface can be reached from any other point traveling on this surface. However, these surfaces end where buildings begin and at the external city boundaries. Thus, the urban access surface has a number of internal boundaries, b, one for each city block and another for city limits, for a total of $B=b+1$. We prove (Section SD) that such a surface is topologically equivalent to a disk with b perforations (or a sphere with B disks removed). It follows that all urban access systems with the same number of blocks, b, are topologically equivalent and share an invariant number, the Euler characteristic, $\chi = 1 - b$, which is independent of geometry.

The topology of *places* depends on their organization relative to each other and to the surrounding access system. To see this, consider city blocks with very different geometries (Fig. 1A-D). The distinctive feature of Figs. 1B and C showing typical blocks in New York and Prague (Fig S4), is that, despite different geometry, all places are adjacent to streets. This is a general property of developed cities and, when it applies, we call the block *universally accessible*. We use graph theory methods (Section SE) to prove that the space of universally accessible places is isomorphic to its access network. This is an intuitive result that follows from the fact that each accessible place has a path or street uniquely dedicated to itself, which in turn is the basis of several urban scaling relations, e.g. between infrastructure volume and built space and city population size (Section SH).

It follows that cities that are universally accessible share the topology of their access systems (Section SF). This result reveals the general topology of cities provided all their places are accessible. The Euler characteristic gives a topologically invariant measure of city size and expresses a surprising type of self-similarity between urban spaces, where sections of one city can be mapped to entire towns provided they share the same number of blocks. These results establish formally the universal character of urban built spaces and show how they can be transformed without loss of function.

However, the structural transformations that render each block universally accessible should not be taken for granted, especially as cities develop quickly. To analyze this issue we represent the *space of places* in each block by its own graph S_0, where edges represent the boundaries of each parcel and nodes their intersection. Each interior face in S_0 represents one parcel (21) (Fig. 1A). This graph reveals complex networks whose topology is not immediately apparent. This process of graph construction, taking faces for nodes and their adjacency for edges, can be repeated successively resulting in new graphs called weak duals (18, 19) (Section SG). The weak dual of S_0 is a graph where each parcel is a node and where physical adjacency becomes an edge. We denote this new graph by S_1, S_1's weak dual by S_2 and so on.

These graphs reveal the topology of places within city blocks through the presence of loops. The block topology is simple when the second weak dual, S_2, is a tree (no loops), Fig. 1. We prove in Section SG that this is a necessary and sufficient condition for blocks to be universally accessible.

Fig. 1D shows a block in Harare, Zimbabwe, (see also S3-5) that does not share these properties. If higher order graphs S_3, S_4,…, S_k are not trees then the neighborhood has several layers of internal parcels: to access parcels in the S_k graph from the outside one needs to cross at least $(k-1)/2$ internal boundaries (Section SG). This situation is characteristic of informal settlements in many of the world's urban slums (22–24) (Fig. 2, S5-S8).

The lack of spatial access is a major obstacle to individual mobility, the provision of emergency services and (physical) neighborhood development (water, sanitation, *etc*) in most developing cities, a situation currently affecting about a billion people worldwide (25, 26). In such situations, providing public services to an informal settlement is prohibitively complicated, expensive and slow (27). However, this is made possible and relatively straightforward after the neighborhood becomes universally accessible, an operation known as *re-blocking*.

Optimal re-blocking can be generated algorithmically by successively minimizing $k>2$, for which S_k is not a tree, under the constraint that a minimal length of accesses is introduced into the neighborhood. This constraint implements minimal disruption and construction costs (4), see Figs. 2D-E. Then, there is only one access solution that achieves the desired topological transformation (Section SE).

We can also allow the constraint of absolute minimal new access length to be relaxed. Then, a (much) larger number of re-blocking strategies become possible. These can be sampled statistically from an ensemble of access configurations characterized by a total mean access length, which plays a role analogous to temperature in statistical mechanics (28). The possibility of generating diverse proposals for optimal re-blocking is important because many local factors play a role in deciding implementable solutions, including the existence of incipient accesses and other forms of informal land-use.

Re-blocking configurations obtained under strict length constraints are shown in Figs. 2D-E. We observe that new infrastructure segments often appear as cul-de-sacs (a proof is given in Section SG), because tree graphs minimize the number of edges necessary to connect a number of nodes and thus new road length. Similar configurations result from minimizing energy dissipation in fluid flows, a principle used to derive optimal transportation networks in river basins (10, 11) and vascular systems (12, 13).

A proliferation of cul-de-sacs is not typical of most neighborhoods because it results in long distances over the network between places that are spatially close. The problem of reducing travel costs (time, energy) between any two places gives rise to a second optimization (8, 29, 30) problem, Figs. S7-8. Travel costs can be reduced by proposing *additional* paths that typically bisect the block along adjoining cul-de-sacs, Fig 2F. The gradual transformation of path systems with many cul-de-sacs to new blocks agrees qualitatively with historical sequences of neighborhood development (1, 8, 9) (Figs. S4, S9).

This geometric optimization has a continuous cost-benefit structure so that more or less road surface will be introduced depending on the desirable ratio of construction to travel costs. Thus, once re-blocked, universally accessible neighborhoods can change shape continuously in response to evolving preferences, new technologies or socioeconomic conditions.

These morphological transformations are typical not only of cities but of other complex systems where transport is mediated by networks. While topology is associated with necessary function, such as the ability of a locus of precipitation to flow to the ocean (11) or of blood to reach every cell in an organism (12, 13), such networks and the places that they serve can be arranged in a continuous spectrum of shapes associated with different physical dimensions (length, volume) and energy budgets.

For cities, once each neighborhood becomes universally accessible, all cities become topologically equivalent (up to number of blocks). This means that buildings and infrastructure networks can be re-shaped continuously as a city evolves without loss of function. It also means that any section of one city, with the same number of blocks, can be deformed onto another. In this way parts of Baghdad can be reshaped into Beijing and *quartiers* of Paris can be deformed into New York City blocks, just like different river basins or individual vasculatures vary in their detailed geometry but display the same essential universal topology.

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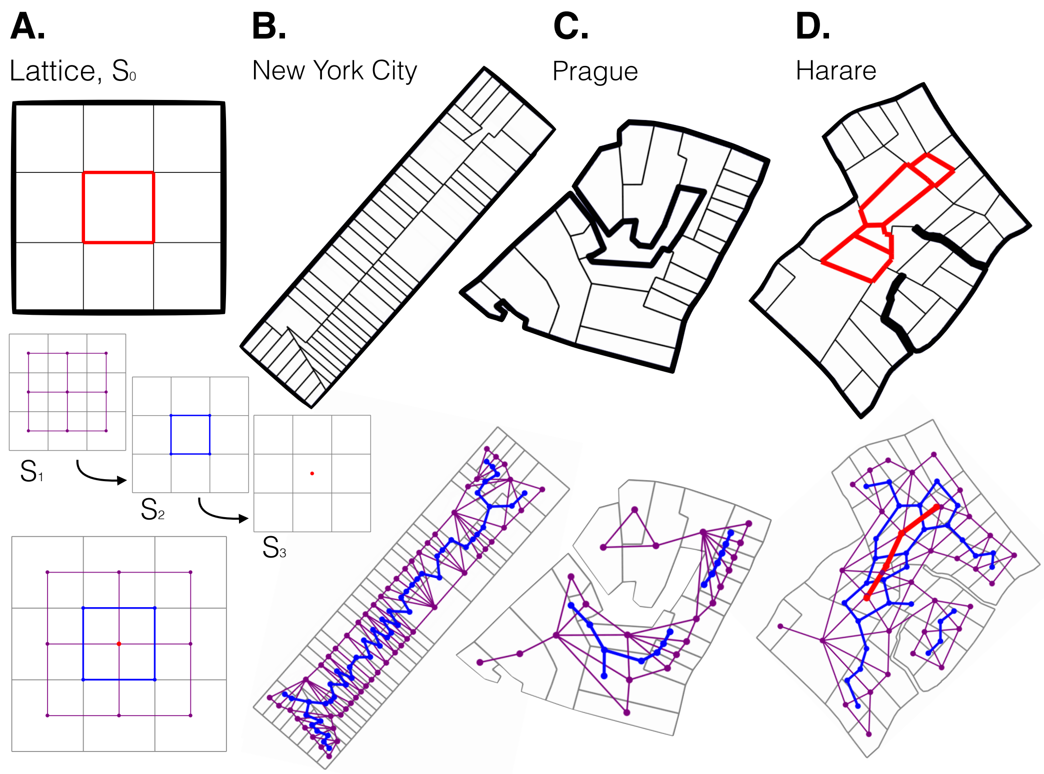

Figure 1: The Topology of City Blocks. **A:** A schematic city-block layout with a single interior parcel (red). The space of places is divided into unitary parcels. This demonstrates the construction of successive dual graphs, S_0 to S_3 (S_1 in purple, S_2 in blue, S_3, a single point, in red). The lower panel shows the four graphs spatially superposed, in the layout of the empirical examples in B-D. **B:** A single block in New York City, showing the city's regular geometry and topology. Each parcel has direct street access so that its S_2 graph is a tree and its S_3 graph is trivial (lower panel). **C:** building layout for a single block in Prague (Czech Republic) in 1842. Nested dual graphs (lower panel) demonstrate that the S_2 graph is a tree and the S_3 graph is trivial, despite the block's complex geometry. **D:** A single block in the Epworth informal settlement in Harare, Zimbabwe. The lower panel shows the block's complex parcel topology: its S_2 graph is not a tree, and its S_3 (red) is not trivial. Data sources are described in Section A in *Supplementary Material*.

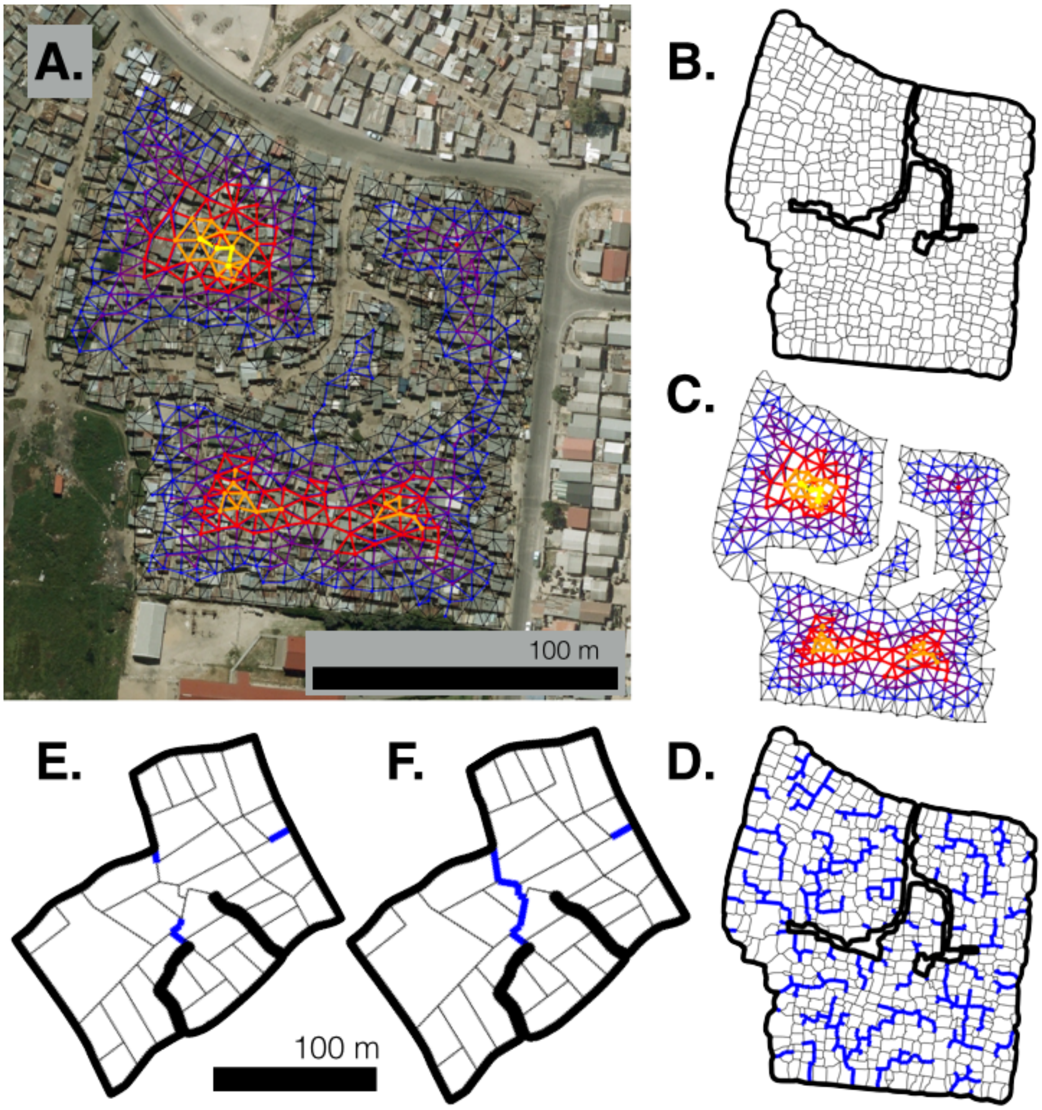

Figure 2: Access System Optimization and Neighborhood Re-blocking. **A:** The odd dual graphs for a single block in the Khayelitsha township of Cape Town (South Africa), shown overlying a satellite image (from 2011). **B:** The S_0 graph displays the block's complex parcel geometry. **C:** Superposed weak dual graphs, from S_1 through S_{11} (blue to yellow) emphasizing the high degree nestedness of many of the block's parcels. **D:** A minimal re-blocking solution that provides access to each parcel. **E:** The block of Fig 1D with a minimal solution to the re-blocking problem (new accesses shown as blue lines.) **F:** More typical neighborhood layout resulting from a process of gradual geometric optimization, showing one additional road bisecting the block. Such solutions increase construction expenditures beyond rendering the block universally accessible but lower travel costs, see Sections SB-C, and Figs S7-8. In these ways, any neighborhood can evolve its spatial form gradually, once its topology becomes *universally accessible*. Data sources and methods are described in Section A in *Supplementary Material*.

Supplementary Materials

The Topology of Cities

Christa Brelsford[1], Taylor Martin[2], Joe Hand[1], Luís M. A. Bettencourt[1]

[1]Santa Fe Institute, 1399 Hyde Park Rd, Santa Fe NM 87501, USA.
[2]Department of Mathematics, Sam Houston State University, Huntsville TX 77341, USA.

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Materials and Methods

Supplementary Text

Supplementary Figures

Materials and Methods

A. Data Sources, Map Digitization and Geo-referencing

We assembled a large and diverse set of detailed urban maps in order to demonstrate that the topological characteristics of cities, described in the main text and detailed below, hold in general, historically and for contemporary cities, across very different levels of socioeconomic development, distinct cultures and geographies.

To do this we assembled a large corpus of detailed urban maps based on historical cadastral records, modern tax assessor maps, community generated structural-level maps and digitized and geo-referenced satellite imagery. Because of space constraints only a few of these are shown in the main paper. Here we provide additional details concerning sources and the creation of geo-referenced digital maps.

The data used to create Fig. 1B is based on the New York City Digital Tax Map (*31*). Data for the parcels, blocks and roads for the entire city are available in digital geo-referenced form. The map of Las Vegas, NV, in Fig. S2 is based on the Clark County Tax Assessor's records, available for academic use through a public records request. This type of map is increasingly common in developed nations, even though many such records are not in the public domain and must be obtained from local authorities on a case-by-case basis.

New kinds of open source mapping data, such as OpenStreetMap, are becoming increasingly common data sources for large scale comparative studies of urban form (*7*, *9*). At this time, the OpenStreetMap community has not reached a consensus on whether parcel level data should be included on the platform. This is because of the many heterogeneous data sources and concerns about parcel data validity and quality. For example, parcel layouts change very quickly in rapidly growing cities, and so the maps may quickly become outdated or inaccurate. There is also no consensus around what information should be attached to a parcel (*32*).

The data shown in Fig. 1C and S4 were obtained from one of the earliest extensive parcel-level maps of Prague (Czech Republic) completed in 1842 (*33*). It was scanned into digital form and geo-referenced by the City Development Authority of Prague (Útvar rozvoje hl. m. Prahy) in 2012. A small part of this map is shown in Fig. S4, overlaid with our identification of the parcels for a single block, as shown in Fig. 1C. Courtyards that are not accessible by road are included in the space of each individual parcel, while open spaces that are accessible by road are included in the access systems. Parcel delineations for major buildings and roads that were constructed before 1842 and are still extant today were used to verify the scale and geo-reference of the original map.

The data for Figs. 1D, 2E, 2F, S5- S7 refer to an informal settlement in Epworth, a suburb of Harare, Zimbabwe (*34*). Maps and data were collected and created by neighborhood resident community members in collaboration with the *Zimbabwe Homeless People's Federation* (*35*) and *Dialogue on Shelter* (*36*), which are federations of *Slum/Shack Dwellers International* (*37*), who also geo-referenced and digitized the map. These organizations have developed and adopted a now widespread process in developing cities of collecting various types of detailed neighborhood data in order to facilitate a community-driven household enumeration and re-blocking process, see Section SI, below. A larger portion of the neighborhood map is shown in Fig. S6. We verified these maps in terms of geo-referencing of parcels, blocks and roads against satellite images.

The parcels in Khayelitsha, a neighborhood of Cape Town, South Africa (*38*) shown in Figs. 2A-2D, S8 were identified based on manual digitization of structures visible from satellite imagery (*39*) following a site visit by our team (including two of the authors, JH and LB) in June 2014 in collaboration with a data collection exercise by *Community Organisation Resource Centre* (CORC) (*40*). During this visit we acquired detailed knowledge (and data) of neighborhood boundaries, services and access conditions.

In addition, we have obtained, created or digitized and geo-referenced many other maps of formal and informal neighborhoods across the world, including in large dense cities such as Rio de Janeiro and São Paulo (where city authorities have mapped most *favela* boundaries), Mexico City (from research surveys), and Kampala, Lusaka, Nairobi and Mumbai from a variety of local sources.

At present, many different methods are becoming available for generating parcel level maps of a city and neighborhood, from those created by resident communities and non-governmental organizations (NGOs) to official cadastral maps, e.g. linked to property records and taxation. Maps created by manual inspection of remote sensing imagery at adequate resolutions or via machine learning techniques (*41*) are also increasingly common because of new data and new analysis tools.

The convergence and cross-verification of all these methods, much facilitated by geo-referencing of data, are making possible new advances in our detailed understanding of the changes of physical space in cities in relation to people's socioeconomic condition and its transformation. Our emphasis here is to demonstrate how some of these practical possibilities require a systematic formal (theoretical and mathematical) understanding of these transformations. The methods introduced in this paper are capable of responding in general and flexible ways to parcel level maps from almost any source and make sense of their topology and geometry through comparative quantitative analysis.

B. Topological Optimization: Minimal Neighborhood Re-blocking

Fig. 1C shows a single block from Epworth (Harare, Zimbabwe) with four interior (not accessible by road) parcels highlighted in red. Existing roads encircle the block. Fig. 2 shows a similar problem for a much larger block in Khayelitsha (Cape Town, South Africa), with 547 parcels, of which only 151 have direct road access. We define a problem of *topological optimization* by asking how to provide each parcel with access using the smallest amount of infrastructure (path length).

We define this *topological optimization* problem in two different ways. The first (*strict* optimization) is easier to understand and formulate but is too rigid for practical use. The second (*statistical* optimization) is more flexible and can be the basis for practical neighborhood upgrading tools discussed in Section SJ, below.

The *strict optimization* approach seeks to find the absolute smallest amount of new paths that need to be built to provide each interior parcel with access. In practice, there is always a unique solution to this problem.

Given a geo-referenced parcel map, the strict optimization problem is solved algorithmically by identifying the shortest path from each interior parcel (along parcel boundaries) to the existing road and measuring its geometric length. While optimal piecewise – parcel-by-parcel this identifies the smallest path length – this strategy may not be globally optimal when there are many internal parcels that are adjacent. This is because there may be solutions with shorter paths when two or more parcels are considered together rather than individually.

This problem of strict optimization can be pursued further, but typically leads to a search space that increases combinatorially (exponentially or faster) as pairs, trios, *etc*, of parcels are considered together. Though feasible for small blocks, this becomes slow or intractable for larger ones. Above all, building the strictly shortest set of accesses is likely not to be viable because of other local considerations, which the algorithm cannot incorporate. These have come up in our discussions with resident associations and city agencies, especially in Mumbai, India. As such, we next emphasize the generation of an ensemble of possible access networks that can be discussed and edited by policy makers and residents, thus defining a *statistical optimization* problem.

The *statistical optimization* problem seeks to identify a set of path configurations that includes the solution of the strict optimization problem but also optimizes for other statistically small amounts of infrastructure introduced into the block. Because there may be many configurations of paths that are not strictly the shortest but that may have other advantages, this leads to a more flexible and realistic set of possible solutions.

In this way, we create a sample of solutions from a statistical ensemble of spatial configurations that render the block universally accessible. This optimization problem makes use of methods of statistical physics by defining a statistical distribution of neighborhood path configurations and sampling it using well-known Monte Carlo techniques of importance sampling (*42*).

To solve this second problem, a statistical ensemble of possible solutions is built in the following way: i) we define a probability distribution over paths, $p(l)$ where l is a path, which makes each path that solves the topological optimization problem more or less likely as a function of its characteristics (cost); ii) we choose a number of paths that connect each interior parcel with the outside and, thus, render the block universally accessible.

In practice, we make path cost a function of path length. Thus, shorter paths that require less infrastructure are more likely. Other properties that contribute to relative path cost can also be included in defining this probability, if desirable.

We have generated a simple algorithm that implements this strategy: For each parcel we identify (strict optimization) the shortest path that makes it accessible. We then further construct n_l additional short paths with the same property. This set of $n_l + 1$ short paths is identified for all interior parcels. In all practical examples developed here we chose $n_l + 1 = 10$. The choice of ten paths per interior parcel is arbitrary: a greater number can be chosen at larger computational cost and vice-versa.

Out of the set of sets of short paths for all interior parcels, a single path is selected with a probability, p. We experimented with different functional forms of $p(l)$, which penalize to a greater extent longer paths. In Figs. 2, S5, S7 and S8, we show results for $p(l)$ inversely proportional to path length squared, $p = \frac{1/l_i^2}{\sum l_i^2}$. This is equivalent to $p(l) \sim e^{-2 \ln l}$, which thus penalizes long paths only gently, as a polynominal (square) function of the *logarithm of path length*. In this context, the power of the logarithm (two, in this example) plays the role of an (dimensionless) inverse temperature in statistical mechanics (*28*, *42*).

Once selected, the path is converted from an interior edge to an access and the set of interior parcels is updated to reflect the new access layout. This path selection process is repeated until no interior parcels remain, and the S_2 graph for the block becomes a tree (see Section SG, below).

Fig. S5 shows an instance of the iterative road construction process for the block shown in Fig. 1C. Fig. S6 shows the actual neighborhood geometry before and after a community driven re-blocking process.

C. Geometric Optimization: Travel Costs vs. Road Construction

We have explained in Section B how different access configurations can be generated statistically to solve the *topological optimization problem* of granting access to each parcel in a city block.

Here, we formalize an additional *geometric* optimization problem that among possible solutions to the topological optimization problem can choose path configurations that reduce travel distance. Note that this second optimization problem deals with improving performance along a gradation of travel costs, whereas topological optimization is necessary and discrete: a city block is either universally accessible or it is not.

To consider the issue of travel distance we introduce the following definitions. Consider P the set of all accesses within a block, including all roads and paths. For each block there is a minimum path length $l_{\min}$, necessary to solve the strict topological problem. There is also a maximum path length $l_{\max}$ that would connect parcels all-to-all. In practice, there will be an actual length of paths between these two extremes, l.

We also introduce a (construction) cost per unit length of path, c. Thus, the total cost of all paths in P, C, is $C=c\ l$. In this Section, we show how a set of paths can be found for each block through the solution of a geometric optimization problem.

To do so, we define a travel cost matrix, T_{ij}, whose entries are simply the distances between each two parcels i and j along the shortest path on the access system. This matrix is traceless and symmetric and all its entries are positive. Note that $T_{ij}(P)$, is a non-trivial function of the block parcel configuration and its set of paths, P. Thus, the travel cost matrix is a complex object that in general needs to be evaluated computationally for each block configuration. Note also that the average travel distance between parcels in the block, $\bar{T}$, can *always* be reduced via the introduction on new roads and paths in P. As a consequence the variation, $\Delta\bar{T}/\Delta P \leq 0$.

We now formulate the problem of finding the optimal block access network, P^*, that minimizes average travel costs, $\bar{T}$ under certain constraints. The simplest realization of the problem is to perform a cost-benefit analysis of introducing new paths. We write the target function, Γ,

$$\Gamma(P) = \bar{T}(P) + \lambda C(P),$$

where the first term measures (decreasing) average travel cost and the second (increasing) construction costs. The constant λ is a dimensionless parameter whose significance will be discussed below. Upon introduction of new paths we write the variation in cost-benefit as

$$\frac{\Delta\Gamma}{\Delta P} = \frac{\Delta\bar{T}}{\Delta P} + \lambda\frac{\Delta C}{\Delta P}.$$

We can now develop a criterion for stopping building new paths. The simplest is to build paths so long as the variation remains negative; that is

$$\frac{\Delta\Gamma}{\Delta P} < 0 \rightarrow \left|\frac{\Delta\bar{T}}{\Delta C}\right| > \lambda.$$

We see that as long as the marginal benefit of building paths exceeds the chosen parameter λ one should continue to introduce more paths. If we assume that the costs of building paths are proportional to length then $\Delta C = c\ \Delta l$. We can write this condition as

$$|\Delta\bar{T}| > \lambda c\ \Delta l.$$

Elaborations of this problem are simple to formulate. For example, under a fixed budget, C_{max}, $\Delta C = C_{max} = c\ \Delta l_{max}$, so that only a certain length of paths will be built. If we prefer to specify a target in terms of minimal acceptable travel cost, then we obtain a constrained optimization problem where λ is not extrinsically specified, but determined instead by the current and target average travel costs and the length paths to be introduced.

In practice, the travel distance between each pair of parcels in a block is measured as the geometrically shortest path between the two parcels along network edges that are paths in the access system, not parcel boundaries. This assumes that travel can only occur over roads and paths, a situation typical of dense urban neighborhoods. The measured minimum travel distance between any two parcels in a block is shown as a matrix where parcels are ordered according to a hierarchical clustering procedure (*43*, *44*). This groups together parcels that are connected by short travel distances, and appear in the matrices of Figs. S7-8 as darker blue matrix blocks. Parcels that are distant appear as orange and red entries in the same matrix. The latter are the primary target distances to be reduced by the geometric optimization described here.

Fig. S7 shows the neighborhood layout and the matrix of parcel-to-parcel travel distance, in meters, for several possible path configurations for Epworth.
After the minimum conditions for solving the topological optimization problem are met, additional path segments could be proposed that would reduce travel costs within the block, quantifying the tradeoff between travel cost and re-blocking effort vs. some desirable target, λ.

The solutions to our topological optimization problem have many more cul-de-sacs than typical urban street patterns. This is the result of *theorem three* in Section G, below. Thus, connecting different cul-de-sacs created by the topological optimization problem is a natural way to identify good choices for additional paths that while short substantially reduce travel distances.

For the example of Figs. 1D, S5-7 (Epworth), it is notable that the addition of a short segment of road connecting two cul-de-sacs effectively splits the city block in two (bisection). This operation meaningfully decreases travel cost, especially by reducing the travel distance between the most distant parcels. This is a general property of connecting cul-de-sacs that provide access to parcels that are close in space but not over the network.

Fig. S7, panel A shows the solution to the topological optimization problem; panels B – D show the additional consideration of geometric optimization where several additional paths have been introduced that bisect the block along existing cul-de-sacs. The centroids of the two most distant parcels in this neighborhood are 205m apart, significantly smaller than the maximum on-road travel distance of 401m in the minimally connected case, and still smaller than the maximum travel distance of 295m in the case with the addition of a bisecting path. The mean travel distance decreases from 151m in the minimally connected case to 116m after the bisecting path is introduced.

Fig. S8 shows the travel cost matrix for the Khayelitsha (Cape Town) neighborhood. In both cases, there are blocks of parcels that are physically distant and the travel distance cannot be minimized beyond a certain amount. Nearly all of the travel distance gains from adding a bisecting path accrue to the parcels that were previously on a cul-de-sac, and now are on the bisecting path.

Supplementary Text

D. The Topology of Access Systems

As introduced in the main text, we can systematically account for all urban built space by dividing it into two categories: 1) the *Access System*, including all roads, and paths; and 2) *Places*, which include buildings and private and public spaces. In this section, we discuss the *topology of the access system*. The relationship between these two spaces and a proof of their topological equivalence is given in Section SF, while the general topology of places within city blocks is characterized in Section SE, with proofs of theorems given in Section SG.

The urban access system is the total physical volume of paths and roads in any given city. A schematic example is shown in Fig. S1, panel A, and a small part of Las Vegas' urban access system is shown in Fig. S2.

The access system of every city is always a compact, connected, orientable, 2D surface. *Connected* means that any point on this surface can be reached from any other point by traveling on the surface. *Orientable* means that there is a global unambiguous definition of up from down. This means that all real access systems in cities are never like familiar non-orientable surfaces, such as a Möbius strip. *Compact* is a somewhat more abstract concept that refers to the topological finiteness of a surface (*45*). In practice, a surface is compact if and only if any triangulation uses a finite number of triangles. This means in practice that we can imagine dividing the total access surface of any city into a number of adjacent triangles. We can do this using a smaller or larger number of triangles, but in any case it is clear that one would need only a finite (if possibly very large) number. Finally the 2-dimensional (2D) character of the access system is obvious in that we can only move on road and path surfaces: we cannot use the underside of these pathways.

Moreover, the urban access system is a 2D surface with *boundaries*. Unlike familiar compact closed surfaces, such as the surface of a sphere or of a torus, the urban access system has both internal and external *boundaries*: On the outside, there is a boundary that traces city limits. On the inside, there is the boundary between accesses and each city block, which consists of *place*s. A schematic representation is shown in Fig. S1.

Given these five characteristics of the urban access system (compact, connected, orientable, 2D, boundaries) we can now characterize its topology using well-known mathematical results.

The only less familiar aspect of the access surface relative to classical examples in topology is the existence of boundaries. The topology of surfaces with boundaries is analyzed with reference to a corresponding surface without boundaries, effectively by plugging each hole with a disk. Then, the resulting surface without boundaries can be analyzed using classical results: the two surfaces are related by keeping track of the number of boundary components. For a city with b blocks it is easy to count boundary components: there are *b internal boundaries* and *one external boundary* so that the total number of boundary components, B, is $B=b+1$.

Analogously to the entire city we define a city *subsection* as a set of contiguous city blocks and surrounding access system, including an external boundary that defines the physical limits of the subsection. When a subsection includes all blocks it is equivalent to the city, and shares its entire access system.

We now formalize our procedure, step by step. Our ultimate goal in this Section is the derivation of a topological invariant that characterizes any given city and establishes the topological equivalence between the access systems of different cities and/or city subsections.

First, we state the following theorem:

Theorem: Topological Classes of Urban Access Systems
The access system of any city with b blocks is topologically equivalent to a sphere with $B=b+1$ disks removed.

Proof: Follows from theorem 4.17 in Kinsey (*46*) in the particular case of the surface being orientable and 2-dimensional since then the access system surface cannot have any "handles" (genus=0).

Corollary:
For any value of b, two cities with b blocks have access systems that are topologically equivalent.

Proof: Follows from the theorem above, since both cities' access systems are topologically equivalent to a sphere with $b+1$ disks removed.

Corollary:
Any subsection of one city with b blocks has an access system that is topologically equivalent to that of another subsection of another city with b' blocks if and only if $b=b$'.

Proof: Define a city subsection as a set of b blocks and an external boundary that defines that subsection in such a way that it is isomorphic to a circle. Then, the theorem above guarantees that the subsection is topologically equivalent to a sphere with $b+1$ disks removed. The two subsections will only be equivalent to each other if and only if they are both equivalent to a sphere with the same number of disks removed. Hence, they will be topologically equivalent if and only if b=b'. See also below.

Corollary:
The access system of an entire city with b blocks is topologically equivalent to a subsection of any other city with the same number of blocks.

Proof: Since the access system of a subsection is the same type of surface as an entire city, they are topologically equivalent spaces if they have the same number of blocks.

Finally, we wish to stress the universal character of the topology of any city access system by characterizing it in terms of a topological invariant. Recall that a topological invariant is defined as a quantity α if, whenever two objects X and Y are topologically equivalent, then α(X) = α(Y).

We can characterize urban access system surfaces in terms of the familiar Euler characteristic, χ. The Euler characteristic, χ, can be applied to objects (complexes) of any dimension(*45*).

For a graph (or 1-complex, consisting only of vertices and edges, where we do not take closed cycles as 2D surfaces), χ is defined as $\chi = v - e,$ where v is the number of vertices, and e the number of edges. For any graph that is a tree, T, a well known result is that $\chi(T) = 1$.

For surfaces (2-complexes), including planar graphs with faces, χ is defined as $\chi = v - e + f$. It can be show that for *any* planar graph, Y, $\chi(Y) = 2$, another well-known result that we will return to later(*19*).

Finally, we would like to compute the Euler characteristic for any urban access system. The final ingredient we need to consider is how to compute χ for a surface with boundaries.

To do this, consider a general surface S with B boundaries. Then define an associated surface S*, where S* is S with all B boundaries patched by disks, each sewn up along the each of the boundaries of S.

We then conclude that the Euler characteristic for the two surfaces are related by

$$\chi(S^*) = \chi(S)+B.$$

This is because, whatever the Euler characteristic for S is, it must be that of S^* minus that of B disjoint disks ($\chi(\text{Disk}) = 1$). Thus, we can compute the Euler characteristic of any surface with boundaries from that of the corresponding S^* surface without boundaries and then subtract the number of boundaries.

This allows us to conclude that the Euler characteristic of an urban access system with *b* blocks (entire city or subsection) is

$$\chi(\text{Access System with } b \text{ blocks}) = 1 - b,$$

where we used the fact that S^* is a sphere and thus $\chi(S^*) = 2$.

To conclude our results we invoke one more theorem [5.17 in Kinsey (*46*)]:

Theorem: *Let S1 and S2 be compact connected surfaces with boundaries. Then S1 is topologically equivalent to S2 if and only if they have the same number of boundary components, both are orientable (or non-orientable), and they have the same Euler characteristic.*

Corollary:
Urban Access Systems are topologically equivalent if and only if they have the same number of blocks.

Proof: This follows directly from the fact that two urban access systems with the same number of blocks have the same Euler characteristic. (This result is redundant with those above, but demonstrates the use of the Euler characteristic more directly.)

These results allow us to conclude very generally that the access system of any subsection of any city is equivalent to another as long as they have the same number of blocks. Each one can be deformed into the other continuously, so that commensurate sets of blocks in Paris, New York City, Las Vegas or Harare, *etc*, are actually topologically equivalent even though they can have radically different geometries.

Finally, we show how the most obvious graph representation of the access system retains the topological signature of its surface, which can be read off from its “graph topology”.

Corollary:
A 1-complex graph representation of an urban access system with *b* blocks, where edges correspond to road and path centerlines and nodes correspond to their intersections,

called the *urban access network, Y*, has the same Euler characteristic as the urban access system, $\chi(Y) = 1 - b$.

Proof: We showed above that $\chi = 1 - b$ for an urban access system with b blocks. Now we show that $\chi = 1 - b$ for the corresponding urban access network, Y.

A 2-complex planar graph always has $\chi = v - e + f = 2$, while the corresponding 1-complex graph has $\chi = v - e$. A 2-complex graph representation, Y_2, of the urban access system that is constructed in the same manner as Y, will have the same number of faces as the urban access system has boundaries. There is one face for each block boundary, and an exterior face that corresponds to the access systems exterior boundary: $f = b+1$.

Then

$$\chi(Y_2) = v - e + f = 2$$
$$\rightarrow v - e = 2 - f = 2 - (b + 1) = 1 - b = \chi(Y).$$

Thus, for any city, the Euler characteristic of the network graph, Y, coincides with that of the access system surface. For this reason, in other parts of this manuscript, we sometimes refer to the topology of the access system surface and the access network interchangeably.

E. The Topology of City Blocks

We now consider a different, and in some ways more challenging, topological problem: the internal organization of city blocks. We will use the general term *parcel* to denote the decomposition of the city block land area into separate units: these are buildings, or more generally, separate land holdings.

The problem of analyzing the topology of city blocks is decomposed into two steps. The first step deals with the relationship of parcels to the access network and is treated in Section F. If a parcel is adjacent to any section of the extant access network then we simply call it *accessible*. A parcel that is not adjacent to the access network is *internal* to the block: its access in practice is mediated through other parcels.

If all parcels in a block are *accessible* we call the block *universally accessible.* As we show in Section F the *topology of these blocks is equivalent to that of the access network*. All blocks are universally accessible in developed cities. For this reason we propose that all cities, as they develop, are eventually made up of universally accessible blocks and thus that the topology of cities is set by that of their access networks. This has consequences for the universal scaling of built space with city size, as discussed in Section H.

The second step deals with non-universally accessible blocks. These blocks are characteristic of dense informal neighborhoods where there was no overall formal planning procedure during construction. This situation is characteristic of many modern day slums; sections of ancient cities also often fall in this category, see Figs. S5-9. Many developed cities once had neighborhoods with many non-universally accessible blocks, see e.g. Fig. S9 for an old section of Toledo, Spain.

We prove in Section SG a set of theorems that characterize the topology of blocks in terms of their *accessible/non-accessible* character. The gist of these theorems is the identification of non-universal blocks, via the graph theoretical analysis of the spatial relationships between parcels. The first theorem gives a necessary and sufficient condition to identify non-universally accessible blocks, even in enormously complicated neighborhoods with many layers of internal parcels, see Figs. 2, S8. It also provides the underlying mathematics for the algorithmic topological optimization problem described in the main text and in Section SB.

The next two theorems establish the minimal number of parcels that need to be crossed to render a neighborhood universally accessible. The second theorem again makes use of graph theoretic concepts by mapping the number of crossings in practice to the stage of a weak dual graph characterizing the neighborhood and thus the number of internal graph loops that need to be opened to make the weak dual graph a tree.

Finally, the third theorem establishes the topology (and geometry) of the minimal length access networks that render the neighborhood universally accessible. In the same spirit of results in efficient transportation networks (see Section I), we prove – in different ways from Rodríguez-Iturbe and Rinaldo (*11*), West et al. (*12*) or Banavar et al (*47*)– that such sections of access networks are *tree graphs*. Thus, minimal-length new path and road segments additions lead to cul-de-sacs. These can be seen in our model solutions, Figs. S5, S7, S8, and in many examples from real neighborhoods, see e.g. Figs. S4, S9.

These results, together with the topology of access networks, establish not only the universal topology of cities but also the mathematics of the spatial transformations necessary for poor and unplanned neighborhoods to develop gradually (*1*, *4*, *48*) (see main text and Sections SB, SC and SJ).

Because of the mathematical nature of these results, algorithms for optimal re-blocking and efficient access can then be readily created and applied to real-world situations.

F. The Topology of Places is Equivalent to the Topology of the Access System

We now discuss the topological relationships *between* places and the access system (roads and buildings). We describe in detail a method for building a continuous invertible map from parcels onto their surrounding paths and roads. We do this by building a graph representation of a city's accesses, a graph representation of the relationship between parcels and roads, and a mapping between the two that proves the they are homotopically equivalent. Homotopy equivalence is a special case of topological equivalence (*45*).

In Section SD, we analyzed the properties of the 2D access system surface and described its topology. At the end of that Section, we collapsed the 2D access system down to the centerlines for the roads and paths that make up the access system, creating a graph representation that we call the *access network*. The access network, Y, is a graph where edges represent roads, paths, or other public rights of way, and nodes represent intersections between them. This is a common way to represent transportation "complex networks" in cities (*49*). In Section SD we showed that the 1-complex access network, Y, corresponding to a given city's access system, has the same Euler characteristic as the access surface and is thus topologically equivalent to it.

In this section we show that the space of places and their spatial relationships is topologically equivalent to the city's access system. This is accomplished by showing that the space of parcels can be continuously retracted (i.e. "shrunk") into the access space. We start with a graph definition of the parcel space.

Definition (bridge graph)**:** We define a 1-complex *bridge graph*, X, representing the relationship of parcels to the access network, Y. Like Y, the bridge graph includes edges to represent roads, pathways and other public rights of way, and nodes to represent intersections between them. We define x as any element (edge or node) in X. In addition to containing all elements of Y, X also contains nodes to represent the centroid of each parcel in the city. This creates a set of initially disconnected nodes N, where each node n_i in N represents a specific parcel in the city. We then add a single edge, e_i, that connects node n_i to the edge or node of $x \in Y$ that the parcel most naturally accesses. If e_i connects n_i to an existing node n_i' in $x \in Y$ no further changes are needed. If e_i connects n_i to an edge e_Y in $x \in Y$, then e_Y is broken into e_{Y1} and e_{Y2} through a process called edge refinement and a node n_i' is added to represent the intersection through which this parcel enters the access network. Then e_{Y1} is an edge that connects nodes n_i' and n_Y.

In this way, X covers the space of Y with edges and nodes representing pathways and intersections between those pathways. X also includes nodes that represent intersections between public pathways and accesses to individual parcels. In this way, the nodes n_i

representing each parcel should be thought of as representing the *intersections* of public and private spaces – for example one's front door.

With this definition it should be clear that *Y* is a subspace of *X*, although some nodes in *X* are contained in the edges of *Y* and some edges of *X* overlap only parts of edges of *Y,* see Fig. S3.

To show that *X* is topologically equivalent to *Y*, we make use of a specific type of equivalence called *homotopy equivalence* (*45*). We note that *X* and *Y* are homotopy equivalent if there exists a continuous deformation of the topological space *X* into *Y* (follows Proposition 7.30 in Lee (*45*)).

We need a definition for the continuous deformation that maps *X* into *Y.* This is achieved through the elementary concept of *strong deformation retraction* in algebraic topology, defined as follows:

Definition: A continuous map *H(x,t)* is a *strong deformation retraction* of a space *X* onto a subspace *Y* if, for $x \epsilon X, y \epsilon Y,$ and $t = [0,1]$, three conditions hold:

1) $H(x, 0) = x,$
2) $H(x, 1) \epsilon Y,$ and
3) $H(y, t) = y.$

This then, takes us from *X* at "time" t=0 to *Y* at "time" t=1; If we start off in *Y* we stay in *Y* for all time.

With these definitions we can state the central result for this section:

Theorem: The space of *Places* and the *Access Network* of any city are homotopy equivalent for universally accessible blocks.

Proof: If there exists a strong deformation retraction from the bridge graph *X* to the access network *Y,* the two spaces are homotopy equivalent(*45*).

We now construct a strong deformation retraction from *X* to *Y.* First, we define the map $r(x): X \rightarrow X$ by:

$$r(x) = \begin{cases} x \, ; \, x \in Y \\ n_i' \, ; \, x \notin Y. \end{cases}$$

Noting that $\{r(x)\}$ is composed of nodes n and edges e and Y is composed of nodes Y_n and edges Y_e, we can define a second map $f(x)$: $X \rightarrow Y$

$$f(x) = \begin{cases} n \in Y_n \;\; ; \; r(x) \in Y_n \\ e \in Y_e \; ; r(x) \in Y_e \\ n_Y \; ; \; r(x) = n \notin Y_n \end{cases}$$

Then, the map f is a retraction from the bridge graph X to the *access network, Y*. The map r retracts all nodes representing parcels down to their intersection with the access network, and the map f retracts those private/public intersections along to the fully public intersections of roads and pathways in the access network, see Fig. S3 for an illustration. Fig. S3B clearly shows that $r(x)$ retracts X to Y, but there is not complete node to node and edge to edge correspondence. The additional map $f(x)$ shows that $r(x)$ can be further refined to create perfect edge to edge and node to node equivalence between Y and a retraction of X.

We now define the map, $H(x,t)$,

$$H(x,t) = (1-t)x + t\,f(x),$$

and show that $H(x,t)$ satisfies all three conditions for a strong deformation retraction:

Condition 1:

$$H(x,0) \;=\; (1-0)*x \;+\; 0*f(x) \;=\; x$$

Condition 2, $H(x,1) \in Y$:

Based on the definition of $r(x)$ and $f(x)$, all $x \notin Y$ are retracted to n_i'. If $n_i' \notin Y_n$, it is retracted to n_Y in the second step $f(r(x))$, where $n_Y \in Y$. Thus $f(x) \in Y \; \forall \; x \in X$, and so

$$H(x,1) = \; f(x) \in \; Y$$

Condition 3, $H(y,t) = \; y$ for $y \in Y$:

It is then clear from their definitions that $r(y) = y$ and $f(y) = y$, and so $H(y,t) = \; y$ for any t.

Note also that we could have played the transformation in H "backwards in time" as t goes from 1 to 0, thus reconstituting X from Y. In this way, we have built a continuous map from the bridge graph X to the access network Y, defining a 1:1 correspondence between any parcel connected to the city's access network and some part of the access network. This shows that the topology of connected parcels in each universally accessible city block is homotopically equivalent to its access network.

This then allows us to conclude that the space of *places* and their interactions and the *access systems* in a city are topologically equivalent to each other and define the overall topology of any city.

The access network, Y, is likely to have trees entering blocks (cul-de-sacs) and also trees that represent infrastructure connections to more distant cities (interstate highways, intercity railroads, *etc*). As we have seen in the previous section, these trees do not influence the topology of Y, X, or the relationship between X and Y. The key factor that r(x) and $f(x)$ rely upon to distinguish what is retracted or not is whether x or n are element of Y, so any trees in Y do not influence our ability to map X onto Y.

Note also that we could have shown even more explicitly that the graphs X and Y are themselves strong deformation retracts of the full urban space and the access system surface, respectively. The fact that these various surfaces eventually retract to Y shows their topological equivalence.

Together with the results of the previous sections we can now say that *any two cities, as sets of places and access systems, are topologically equivalent if they have the same number of blocks.*

A caveat applies to blocks where not every parcel is connected to the access system. If there exists a parcel represented by n_i that is not directly connected to the access system, no edge e_i will be created. It is then clear that $r(x)$ cannot retract n_i along e_i into the space of Y. Thus $H(x,t)$ is not a deformation retraction from X to Y if X contains parcels that are not connected to the access system.

We address the case of parcels that are not connected to the access system in the next Section.

G. City Block Topological Theorems

Finally, we complete the analysis of the topology of cities by dealing with city blocks that are *not universally accessible*. We use graph theory to prove a set of theorems that show how even blocks with extremely complicated parcel structures can be quantitatively analyzed and rendered connected with minimal disturbance.

Definition: A block S is called *universally accessible* if every parcel within S borders a road. Otherwise, S is not universally accessible.

Definition (minimal set of accesses)**:** Interior parcels can be connected to the urban access system by converting edges in the S_0 graph (see Fig. 1) from parcel boundaries to roads. The *minimal set* of additional roads necessary to connect a given parcel to the road system is the set of edges with the shortest total length such that at least one node contained in the set of edges to be converted is part of an existing road, and at least one node is part of the face in S_0 that surrounds the parcel. The minimal set of accesses is the unique solution of the strict topological optimization problem discussed in Section B.

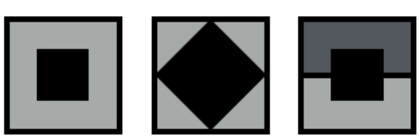
Theorem One:
A block S is universally accessible if and only if its stage-two graph, S_2, is a tree.

Theorem Two:
If a parcel is represented with a node in the S_k graph, at least $\frac{k-1}{2}$ parcel boundaries must be crossed in order to reach it from the nearest section of the access system.

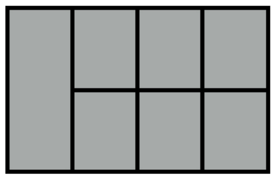
Theorem Three:
There will be no loops in the minimal set of additional roads necessary to connect all interior parcels to a road. Thus, newly constructed roads in the minimal set of accesses form a tree or set of trees (cul-de-sacs).

Proofs:

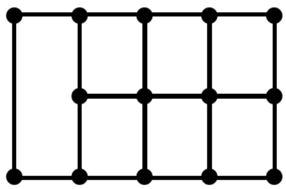
We start by laying down some additional definitions and the procedure of (weak dual) graph construction. Consider a neighborhood, divided into parcels of land separated into connected components of land (blocks) by roads. Here, we will assume that no single parcel of land completely encloses another, no parcel of land touches a road at only points, and no two parcels share multiple non-contiguous borders, as shown below:



Then, given a block S, we assign a *stage zero graph* S_0 where nodes and edges are created to represent the parcel's geometric boundaries (see also Fig. 1). This graph is a planar graph, meaning that edges only intersect at nodes. We think of the graph S_0 as sitting on the surface S. For example for a given block,
$S =$

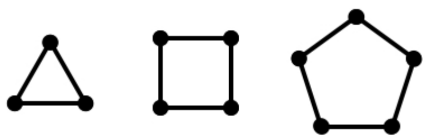


and S_0 =



Definition: In a graph G, a *cycle* is a collection of m vertices and m edges arranged so that each vertex has exactly two edges incident to it, where $m \geq 3$.

For example:



, etc.

As usual, the *degree* of a vertex is the number of edges incident to it.

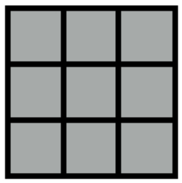
Definition: A *face* of a planar graph is a maximal region in the plane that contains no edge or vertex of the graph.

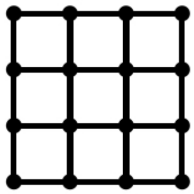
Note that every planar graph has one unbounded, exterior face. Here, we will disregard the exterior face so that each face in S_0 corresponds to a parcel in S.

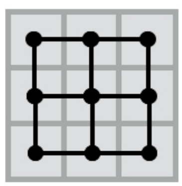
Now, given a block S and a stage zero graph S_0 for S, we can define a *stage one graph* S_1 in the following way:

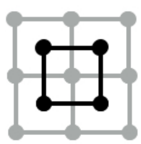
Definition (weak dual graphs): For each bounded face of S_0, we assign a vertex in S_1. Two vertices of S_1 have an edge between them if and only if the faces of S_0 they represent share a common border of at least one edge in S_0. Then, S_1 is the *weak dual graph* of S_0. For a block S, we may then assign a *stage k graph* S_k defined recursively by repeating the process used to construct S_1 from S_0 on the stage k-1 graph S_{k-1}.

For example, this leads to the sequence of graphs (in black, below):

$S =$ 

$S_0 =$ 

$S_1 =$ 

$S_2 =$ 

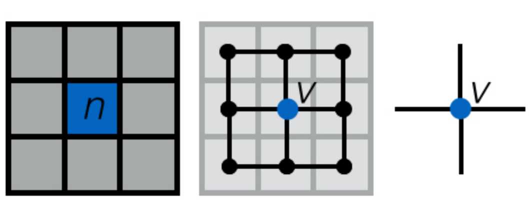
etc. See also Fig. 1 and 2 in the main text.

Definition: A vertex v of a graph G is called an *interior vertex* if there exists a cycle surrounding v so that deleting this cycle from G results in either:

1) Two connected components, one of which contains vertex v and all of its incident edges,

or

2) Just the vertex v and its incident edges, as in:



.

Observation: A parcel n in block S does not border a road if and only if n is surrounded on all sides by other parcels in S. This is true if and only if the vertex v of S_1 that corresponds to parcel n is an interior vertex of S_1.

Definition: A graph G is called a *tree* if G contains no cycles.

Proof of Theorem: *A block S is universally accessible if and only if its stage two graph S_2 is a tree or set of trees***.**

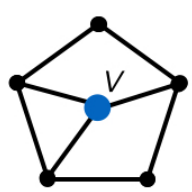
First, we show that if S is universally accessible then S_2 is a tree. We do this by showing that if S_2 is not a tree then S is not universally accessible.

Suppose that for a given block S, S_2 is not a tree. This means that there exists an interior face of S_2 whose boundary is a cycle σ consisting of m vertices, $x_1, x_2, \ldots x_m$ of S_2 and m edges. Each vertex x_i in σ represents a face f_i of S_1, where face f_i shares a common edge with face f_{i-1} (mod m) and face f_{i+1} (mod m). Furthermore, each of these shared edges is incident to a vertex v of S_1 that represents the interior face of S_2.

Thus, the cycle σ in S_2 corresponds to a subgraph of S_1 consisting of the m faces, $f_1, f_2, \ldots, f_m$, arranged in a circle around the vertex v.

Example:

σ in S_2 = 

corresponds to  in S_1.

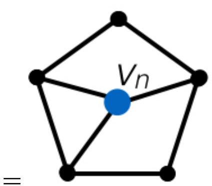
Therefore, vertex v is an interior vertex of S_1, so it corresponds to a parcel of the block S that does not border a road. This shows that block S is not universally accessible.

Now, we will prove that if a block S is not universally accessible, its stage two graph, S_2, is not a tree. We assume that there exists a parcel n of a block S that does not border a road. Thus, there is a vertex v_n of S_1 corresponding to parcel n that is an interior vertex of S_1.

Consider the subgraph V_1 of S_1 consisting of a minimal cycle surrounding vertex v_n, vertex v_n itself and all edges incident to vertex v_n.

Now, we consider the subgraph V_2 of S_2 that represents V_1. V_2 will contain one vertex for each face of V_1 connected by one edge representing each edge incident to vertex v_n.

For example:

$V_1 =$ 

$V_2 =$ 

We conclude that the subgraph V_2 of S_2 is a cycle with m vertices, where m is the degree of vertex v_n in S_1.

This says that the stage two graph of S contains a cycle, and is therefore not a tree. This concludes the proof of Theorem 1**.**

Note that if a stage k graph S_k is a tree, then the stage (k+1) graph S_{k+1} vanishes, as there are no interior faces in S_k, so there are no vertices in S_{k+1}.

Definition (block complexity)**:** From this, we may generalize a *complexity* on the block S as the smallest positive integer k such that S_k is a tree. Every block will be characterized by a positive, discrete value of this complexity. The complexity of universally accessible blocks is $k \leq 2$. Non-universally accessible blocks will have $k > 2$.

The complexity of a block S or a parcel represented in graph S_k is also useful for determining how many parcel boundaries must be crossed to reach an access from an interior parcel. This gives a topological measure of the difficulty of accessing the urban spaces from a given parcel and of achieving universal connectivity for a given block.

Theorem Two: If a parcel is represented with a node in the S_k graph, at least $\frac{k-1}{2}$ other parcel boundaries must be crossed in order to reach an access (road).

Proof of Theorem Two:

For any parcel *n* of block *S*, the minimum number of parcel boundaries that must be crossed in order to reach a road is represented by the minimum number of edges necessary to form a path from v_n, the vertex representing *n* in the S_1 graph, to an exterior vertex of S_1.

Observe that, in the algorithm for creating the S_k graph of a block *S*, parcels of *S* are represented by faces of S_k when *k* is even and nodes of S_k when *k* is odd. Furthermore, for even *k*, if a face of S_k touches an exterior vertex, that face is represented by an exterior vertex in S_{k+1}. Finally, observe that, for odd *k*, parcels represented by an exterior vertex of S_k are not represented at all in S_{k+1}.

Therefore, suppose a parcel *n* requires a path of length *l* to connect vertex v_n to an exterior vertex in S_1. It is clear that in S_3, the path from v_n to an exterior vertex will have length $l - 1$, and so on. The vertex v_n will thus be an exterior vertex of the graph S_{1+2l}. Therefore, we see that, if vertex v_n appears in graph S_k, then $k \leq 1 + 2l$, which says that $\frac{k-1}{2} \leq l$. This concludes the proof.

Theorem Three:

There will be no loops in the minimal set of additional roads necessary to connect all interior parcels to a road. Thus, newly constructed roads in the minimal set of accesses form a tree or set of trees (cul-de-sacs).

Proof of Theorem Three:

We may consider the access network of a given block as a subgraph of the stage zero graph S_0. In order to connect all parcels to a road, we consider parcel boundaries, which are represented by interior edges in S_0. We may then choose a set of such edges of S_0 to represent additional segments of road needed to ensure that the block is universally accessible. There will be several choices for this set of additional roads; we choose the one that has the fewest total geometric length of edges (*minimal* set of accesses).

Suppose that there exists a block for which the minimal set of additional roads is not a tree or set of trees. Let *M* denote the subgraph of S_0 consisting of edges belonging to the minimal set of roads along with the nodes incident to these edges. We are assuming that there is at least one cycle in *M*. Every face of S_0 representing an interior parcel must share at least one node with *M* in order for every parcel to be accessible via existing or new paths. However, all connected planar graphs have a spanning tree, which is a subgraph containing all nodes of the graph but no cycles(*19*). Then, we let *M*' be the subgraph of *M* consisting of spanning trees for each component of *M*. Thus, every face of S_0 representing an interior parcel will share a node with *M*', making every parcel accessible via existing or new roads, but *M*' has strictly fewer edges than *M*, as it is a subgraph containing no cycles. This contradicts the choice of *M* as minimal. Therefore, the set of newly constructed roads must form a tree or set of trees. This concludes the proof.

The result of Theorem 3 explains the proliferation of cul-de-sacs under strict topological optimization, a curious fact widely observed in the development of street patterns in dense old cities (Fig. S9) and in modern informal settlements. However, this pattern – the "cul-de-sac effect" - is often actually desirable as it may promote safety and community cohesion (*50*) and is often observed in designed planned suburban communities, e.g. in Las Vegas NV (Fig. S2).

H. Consequences of City Topology for Urban Statistical Properties

Recently, growing availability of data, especially from digital geo-referenced urban maps and analysis tools, especially from complex network theory, have lead to a growing number of studies to characterize the spatial layout of cities. Here, we relate our results to the main findings in this literature and clarify how the systematic topological analysis of urban built spaces developed here is necessary to make sense of both street layout statistics and the scaling of urban built spaces with city size.

Street Patterns as Complex Networks

One area of intense activity has been the statistical analysis of street patterns, once these are represented as complex networks (graphs). Much like our construction of the network graph, *Y*, city streets are often abstracted from the actual access surface to form a *primal graph* where nodes represent intersections and edges represent centerlines (*8*, *51–53*). Alternatively, the access system has also been represented as a *dual graph,* where nodes represent streets and edges represent street intersections (*21*, *54–56*). This process of dual graph construction is analogous to the first stage weak duals we used to prove the city block theorems of Section G, but clearly used in a different way and at a different scale (block vs. city). The authors of reference (*53*) argue for certain advantages of the primal approach, as in *Y*.

Once these graphs have are constructed based on urban and regional data, many standard complex network properties have been measured, including statistical distributions of street lengths (*55*, *57*), several measures of centrality (*49*, *51*, *58*–*60*), and the distribution of block areas (*9*, *49*, *59*), among others.

In most cases, except in intensively planned layouts (*7*, *9*, *58*), these statistical distributions are very broad (power-laws or lognormals) pointing to the large variability of street patterns and block areas and in some cases even to their apparent scale independence.

Although combinations of these measures have been used to attempt to classify urban shape classes (*7*, *58*, *61*), the large variability makes the *geometric classification* of cities somewhat arbitrary, depending on quantities chosen and cutoffs between categories. This has been known more qualitatively through many attempts to distinguish city geometries (*3*, *62*, *63*) based on street patterns.

Hence, we believe that accepting that urban geometry is variable while the topology of cities, as derived here, is universal provides us with the most fertile starting point to analyze the geography of built space in cities. In particular, this allows us to emphasize the primary character of cities as built spaces that co-evolve with their socioeconomic life.

Geometric Optimization: Construction vs. Connectivity costs

Another branch of the literature considers the fundamental tradeoff between connectivity and construction costs to create generative algorithms for street networks. We discussed this issue in detail in the main text and in Sections B and C, in terms of providing access to places. We have described this tradeoff as the *second* of two optimization problems, occurring *after* universal connectivity at minimal cost has been achieved. The literature of attempting to use algorithms based on these ideas to generate urban and regional layouts goes back at least several decades(*64*–*66*) but recently these algorithms have become from sophisticated in terms of the optimization strategies invoked.

As we have discussed above, real city street networks lie on a continuum of morphologies between the two extremes of tree-like graph structures and fully connected (spatial, planar) networks. These two network extremes constitute the minimum and maximum number of links between a given number of nodes in a connected graph, respectively. Along this continuum one can define measures of network efficiency (*67*), "meshed-ness" coefficients (*8*, *52*) and the balance between the information required to navigate the space and the distance of travel (*49*, *68*) as possible targets of optimization for any street network design.

These measures have been used to consider how streets may be organized in the absence of a central planner, to consider possible principles about how city street networks arise (*69*) or how roads should be laid out to facilitate optimal traffic flow (*29*), often by adapting algorithms from efficient transport networks in other complex systems.

The fundamental ingredient missing from current approaches is the fuller consideration of *places* as the end-points of these networks and, consequently, of emphasis on the function (rather than form) of urban road and street systems in providing access between any two places. Thus, as they stand, these ideas do not yet generalize to a full theory of (built space) in cities. This is because the topological properties of street networks do not yet include full consideration of the relationship of streets to the places they are used to access or to consistent ways to topologically characterize each city and compare cities or parts of cities to each other. All these points have been well know to urbanists (*63*, *70*) and we hope that a future convergence of more realistic optimization algorithms with topological and functional considerations will improve our current ability to tackle these issues in ways that emphasize the fundamental socioeconomic character of cities (*70*–*72*).

Scaling relations for urban space and infrastructure

We conclude this section by demonstrating two additional ways in which the topology of cities allows for the prediction and interpretation of certain quantitative scaling properties of cities. Scaling refers simply to how certain properties of cities co-vary with city size (*73*–*76*). Here, we concentrate on two spatial properties of built space: the scaling of street intersections with street segments and the scaling of urban built area (accesses + places) with city size.

It has been observed that the number of street intersections scales linearly with the number of street segments in primal graphs such as Y. Curiously, this seems to be a property not only of city street networks (*8*, *77*) but also of regional street patterns (*9*). This property derives necessarily from the topology of access systems and of the resulting properties of Y_2 as a planar graph (see Section D) with cycles for each block:

Because Y_2 is a planar graph, $\chi(Y_2) = v - e + f = 2$, where v are vertices (intersections), e are edges and f are faces (land blocks). This means that we can express intersections as

$$v(e) = 2 + e - f.$$

Finally, in a planar graph we can express *f* as a function of *e*. We write *f*=*a.e*, necessarily with *a*<2/3 and conclude that

$$v(e) = 2 + (1 - a)e$$

with 1-*a*>0.

The number of vertices per face in street layouts has been observed to vary considerably (*51*, *59*) However, because of the topological properties of the access system, we can say that this is always a linear relation with the constant a resulting from different statistical mixtures of block configurations. For regular lattices of all kinds a is a constant taking well known values (*78*).

Finally we comment on the scaling relationship between urban infrastructure systems, built space and the population size of cities.

Present and past cities around the world are observed to exhibit quantitatively predictive scaling properties (*74–76*, *79*, *80*). Specifically, urban scaling theory (*76*) predicts that city spatial configurations are the (quantitative) result of balancing out net benefits from socioeconomic network interactions and costs of movement of people, goods and information. In this view, cities are functionally a kind of social reactor that enables a larger scope of human social processes associated with efficiencies, innovation and information encoding with their population size (*3*, *70*, *81*, *82*).

The formalization of these ideas implies that the area of the access system in cities (defined functionally as metropolitan areas, not just as city cores) scales as $A_n(N) = A_0 N^{\gamma}$, where $\gamma \sim 5/6$ in the simplest cases (*76*, *79*). This also implies that average population densities increase with city size with an exponent ~1/6.

But, curiously, this prediction applies empirically to the total *built space* of all large (population > 100,000) cities in the world measured via satellite imagery (*83*), not just their road area. The topological properties of cities derived here help us see why: all cities result from the co-evolution of places and accesses, as stated in the main text and proven formally in Section F. Because of the topological equivalence between these two spaces they must scale in the same way as cities grow (a sort of "holographic principle" for cities) thus explaining the coincidence between the scaling of the infrastructure volume of a city, its total built space, and consequently also the volume of its space of places.

The convergence of all these facts about cities and the generative ability of their formal topological characteristics is, in our opinion, an exciting prospect to understand their shape while simultaneously emphasizing the open-ended character of their built spaces and their functional socioeconomic roles. This is needed to transform simple and static optimization processes into dynamic development dynamics for cities, thus hopefully breathing new (mathematical) life into the processes of spatial evolution in cities and urban systems (*3*, *63*, *70*, *81*, *82*, *84*).

I. Efficient Transportation Networks

A large literature has developed over the last few decades on the network morphology and functional consequences of efficient transport and communication (*10*, *12*, *29*, *47*, *64*, *85–87*, *87–94*). Here we provide some additional context and discussion of the relationship of our present results to this body of work.

Historically, geographers have attempted to understand patterns of transportation networks in terms of the tradeoff between construction and movement costs (*64*, *95*).

In complex systems optimization principles associated with energetically efficient transport (especially of fluids) have provided a framework to model and understand river networks (in 2D) (*10*, *85*, *86*, *90*, *96*) and the large-scale structure of the vascular system of biological organisms (in 3D), and its connection to organism metabolism (*47*, *88*, *89*, *93*) and associated physiological time-scales (*97*).

These model networks exist at an extreme of the space of possible connected transport systems as the consequence of the strict need to minimize energy dissipation in transport. Our results provide a different point of view showing that this type of optimization effectively requires the reduction of the number of transport sections (edges) for a given number of junctions (nodes), see Theorems Two and Three in Section G. As such, optimization problems encoding these principles generally result in tree structures as these network configurations contain the minimal number of links, given a number of nodes.

In other complex systems the situation is the opposite, at least locally. Many real networks seem to reflect the essential need for all-to-all (potential) interactions, or the requirement of low point-to-point transportation costs (e.g. in terms of distance) vs. construction costs, see also Section C. Such networks include urban access systems (*76*), the local structure of brain cortex (*91*, *92*) and certain capillary structures of plants and animals (*98–100*). In these cases the simplest measures of efficiency, in terms of construction costs or energy dissipation per unit length of the network must be balanced by the functional needs for widespread and flexible links between any two places. While high construction costs vs. connectivity leads to hierarchical networks (trees), emphasis on connectivity over construction costs, leads to all-to-all graphs. Most real networks in complex systems must, in fact, strike a balance between these two simple extremes.

In practice, many large systems characterized by all-to-all local connectivity also display long-range connections, which promote system-wide integration. The large-scale structure of connectivity is typically sparser, and these links tend to aggregate processes (highways, nerves) into "bundles" that minimize both occupied volume and length (construction costs) and travel time (transport) (*76*, *101*, *102*). Efficient networks that combine these two types of constraints will show only partial hierarchies, and will not be tree structures(*3*). They are likely also to be more plastic and capable of adaptation and learning as we know qualitatively in cities and nervous systems.

In cities, such complex network structures still preserve a certain fractal structure that reflects the topological block structure discussed presently and are an ingredient to the theory of urban scaling (*76*) that set the spatial properties of all cities (*74*, *76*, *103*), including in history (*79*, *80*).

The brain and nervous systems are even more complex and less understood, but the essential balance between (potential) local all-to-all connectivity and large scale connections connectivity seems to be also essential. It has been argued persuasively that long-range processes organize themselves as nerve bundles, creating a pattern of large-scale architecture with some hierarchical features, that nevertheless is decentralized and not tree like (*3*).

The present paper shows how *necessary* topological properties of access set the basis for more flexible and possibly open-ended issues of morphological adaptation. The great strength of a topological perspective into the architecture of complex systems is that it provides us with large equivalent classes of system form that preserve the same essential function. This, we believe, is a fundamental insight that provides freedom for evolutionary processes to act in flexible and open ended-ways, despite some fundamental physical constrains on the system (*104*). The fuller study of these issues is an exciting prospect but will require substantial future effort across a number of different complex systems.

J. Re-blocking as the Major Strategy for Neighborhood Development

Finally, we want to emphasize the practical importance of the re-blocking processes, discussed above and in the main text, for neighborhood development.

Urbanization trends throughout the world are transforming the nature of issues of human development into primarily urban problems. In most developing cities, informal settlements or slums comprise substantial parts of the population, in many cases its majority, especially in South Asia and Sub-Saharan Africa (*25*, *105*). In total, UN-Habitat estimates that about a billion people live currently in slums (*105*). Slums can be very diverse in their physical appearance, layout, type of location, *etc*, but they do share common characteristics of lack of accesses, lack of urban services, and unplanned land uses (*106*), see Figs. 1, 2, S5-S7 for several examples.

The magnitude of the problem worldwide is enormous, but there is also increasing awareness of this issue and attempts to create the conditions for neighborhood and general urban development (*25*, *26*, *107*, *108*). The past *Millenium Development Goals* (*109*) targeted significant improvements to the lives of 100 million slum-dwellers worldwide (goal 7, target 11, also known as “Cities without Slums”), and the upcoming post-2015 *Sustainable Development Goals* are expected to contain much more ambitious targets (*110*). In addition, substantial programs for informal settlement *upgrading* and re-settlement are in development at the national level (*27*, *111*).

The principal issue to do with neighborhood development is how to affect the greatest positive change most effectively. Increasingly, slum upgrading *in situ* has become a main viable strategy, often preferable to evictions and relocations that too often fail to provide sustainable solutions (*26*, *27*, *111*).

Re-blocking, by providing access to each parcel and building in a neighborhood, is the main physical enabler of any slum-upgrading strategy (*26*, *112*). This is mainly because it facilitates the introduction of urban services and infrastructure, and a gradual process of morphological neighborhood change that eventually may lead to the fusion of small parcels and building upgrades or reconstruction. UN-Habitat currently recommends street focused infrastructure upgrading as a major strategy for neighborhood development that can significantly improve socio-economic outcomes (*26*). The costs of providing services before and after re-blocking vary tremendously, often by a factor of ten or more, making the critical difference between providing a service of not (*113*). This is because urban services (water, sanitation, gas, *etc*) access places preferably following (and buried under) streets and paths.

Re-blocking and concomitant street access to all places in the city block also has a number of important spillover effects (*26*): It allows an incremental approach to neighborhood change, encourages participatory planning via enumeration and community mapping, it improves the physical integration of slum in the city, it assist in land regularization and security of tenure, and leads to higher revenues for the city. Thus (*26*), “streets become tools for social, economic, juridical and spatial integration of slums with the city.*”*

Supplementary Figures

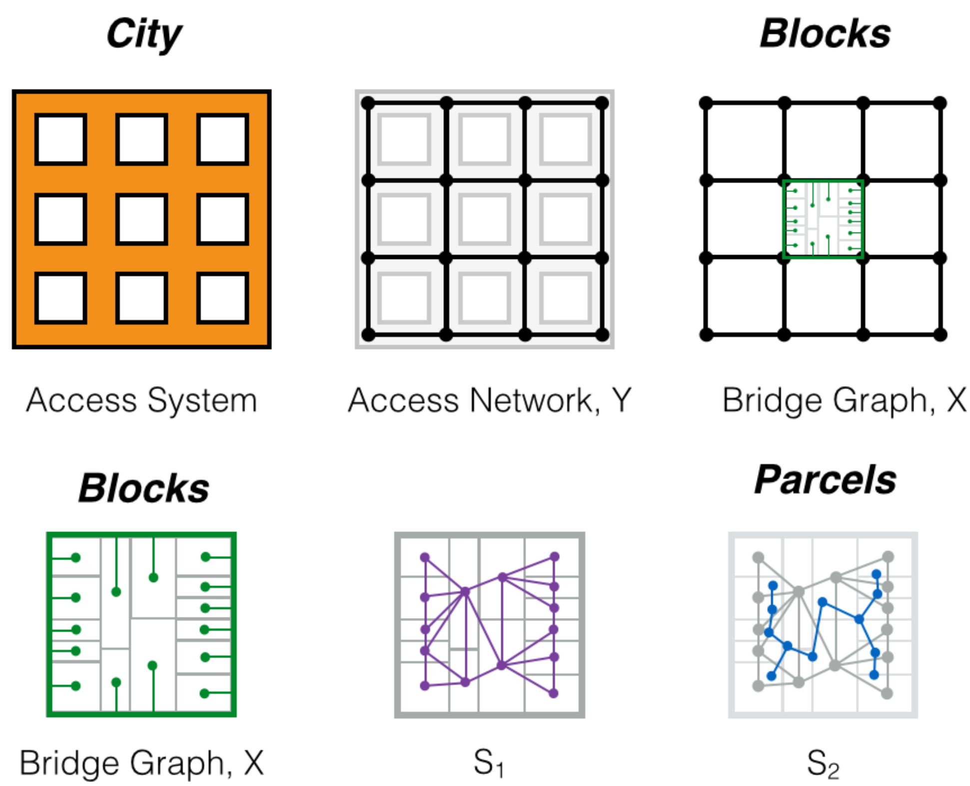


Figure S1. **Topological Constructs for the Systematic Analysis of Urban Topology** From left to right in the top row these panels show how the built space of an entire city can be systematically analyzed in terms of the topology of its access system (orange), its access network *Y* (black), and the relationship from places (parcels) within blocks to their accesses represented as a bridge graph *X* (green). On the bottom row, the very local relationships between each parcel and its neighbors, represented by a hierarchy of S_k weak dual graphs for each city block (purple and blue).

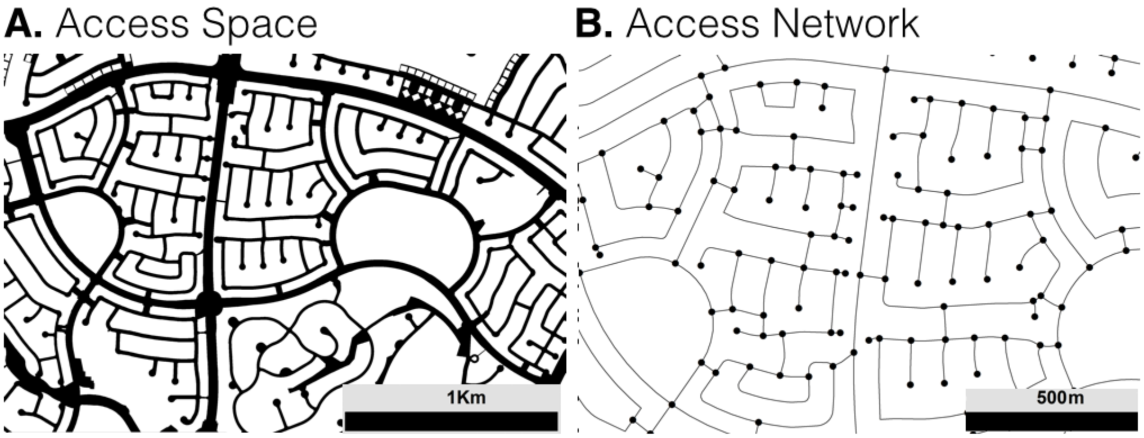


Figure S2. Las Vegas Access System and Access Network
A shows the access system for a neighborhood in Las Vegas, Nevada. **B** shows the access network, *Y*, for the same neighborhood. Data is provided by the Clark County Tax Assessor's Office (*114*). Note the prevalence of designed cul-de-sacs, typical of many planned suburban communities (*50*).

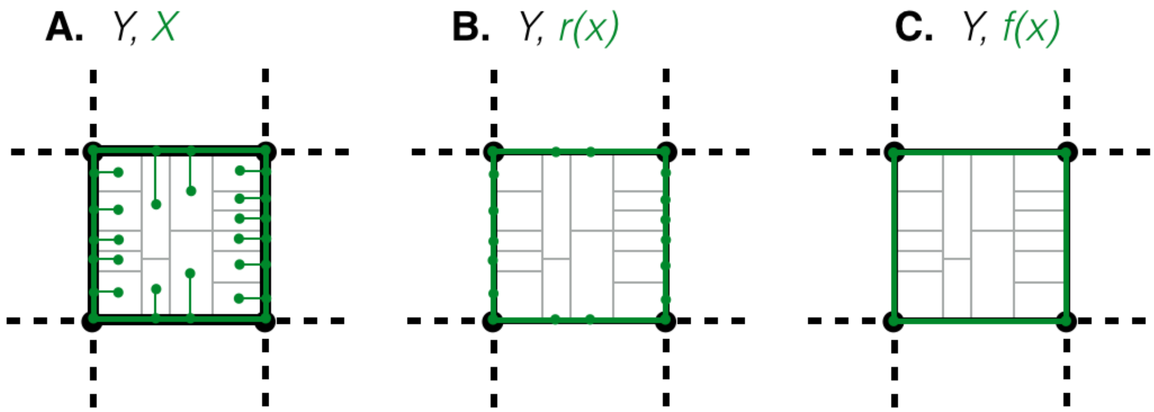


Figure S3. Bridge Graph Retraction

The two-step retraction from the bridge graph *X* (green), to the access network *Y* (black), is shown. **A** shows a subset of the original *X* and *Y* graphs that is a single block. Dashed lines denote the continuation of *X* and *Y* as in Fig. 1, though they are not shown. It is clear from **B** that *r(x)* is sufficient to map all nodes and edges of *X* into the space of *Y*, although there is not a node to node and edge to edge equivalence. **C** shows an additional map *f(x)*, which relies on the map *r(x)* to achieve full edge to edge and node to node equivalence between *f(x)* and *Y,* demonstrating a deformation retraction from *X* to *Y*.

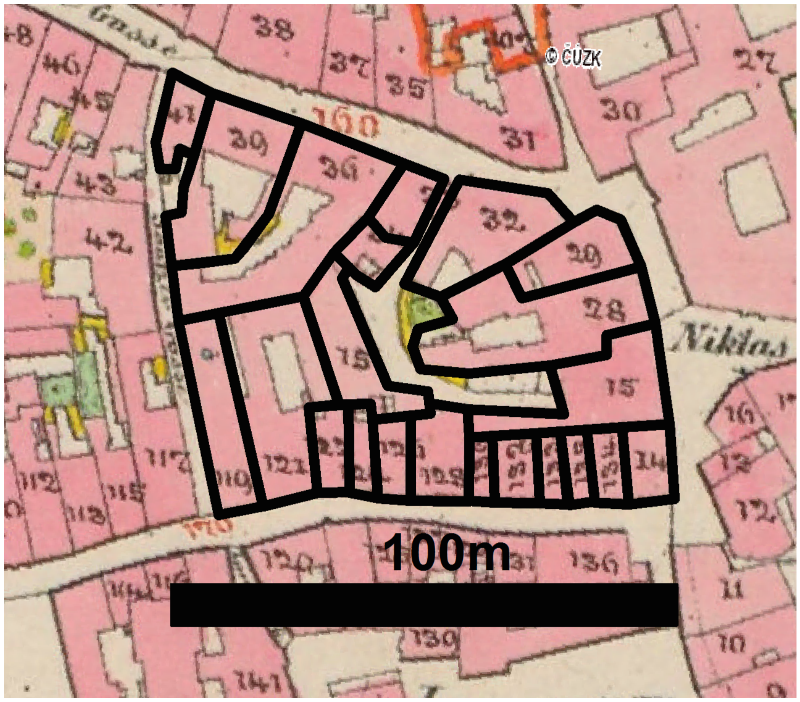


Figure S4. Prague Cadastral Map and block Parcel Layout. The first Stable Cadastral Map of Prague (*33*), surveyed between 1817 and 1840, and completed in 1842. Open spaces and courtyards are included with the parcel they most likely border.

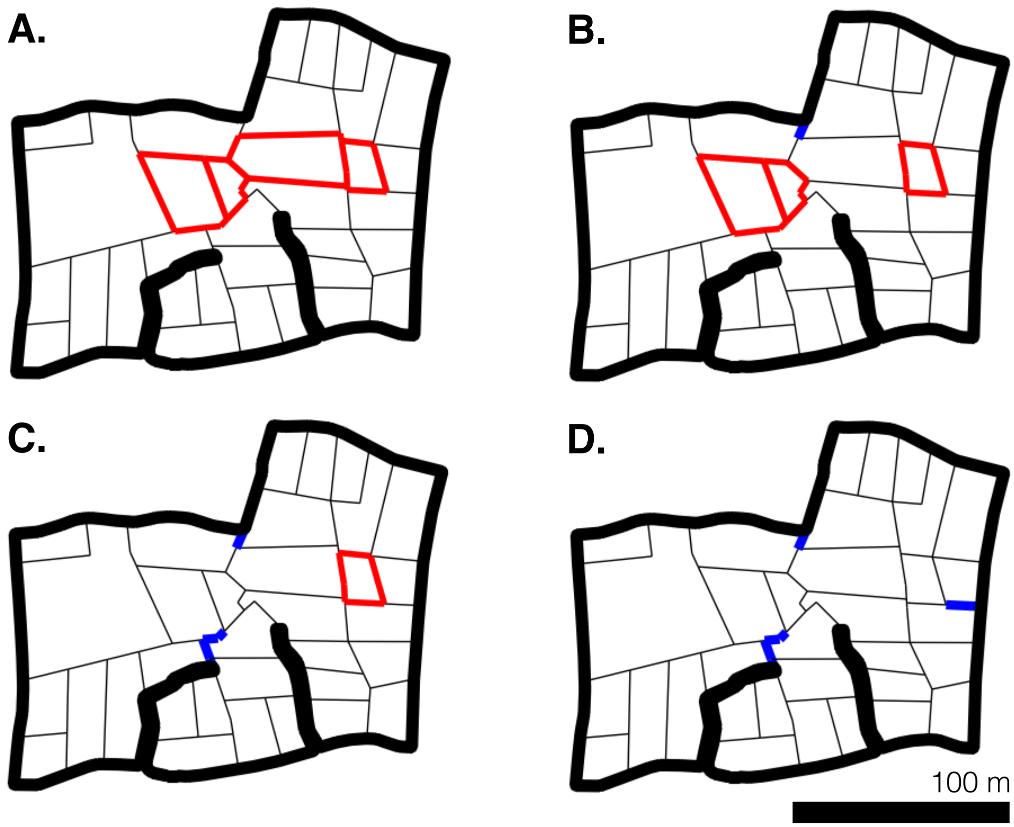


Figure S5: **Epworth (Harare)**(*37*) **Minimal Re-blocking**
AThe example block shown in Figure 1C, surrounded by the original access system (bold black) with four interior parcels (red). **B** One new path (blue) connects a single interior parcel to the access system. **C** An additional path (blue) connects two of the remaining interior parcels to the access system. **D** A third path (blue) connects the final interior parcel to the access system, making the block universally accessible.

Figure S6: Epworth (Harare)(*37*) Before and After Re-blocking
A Many blocks in Epworth in their original layout before re-blocking. **B** The planned outcome of the re-blocking procedure proposed by the resident community. Parcels highlighted (red) do not have direct road access. This community-driven re-blocking process has created access to the vast majority of internal parcels.

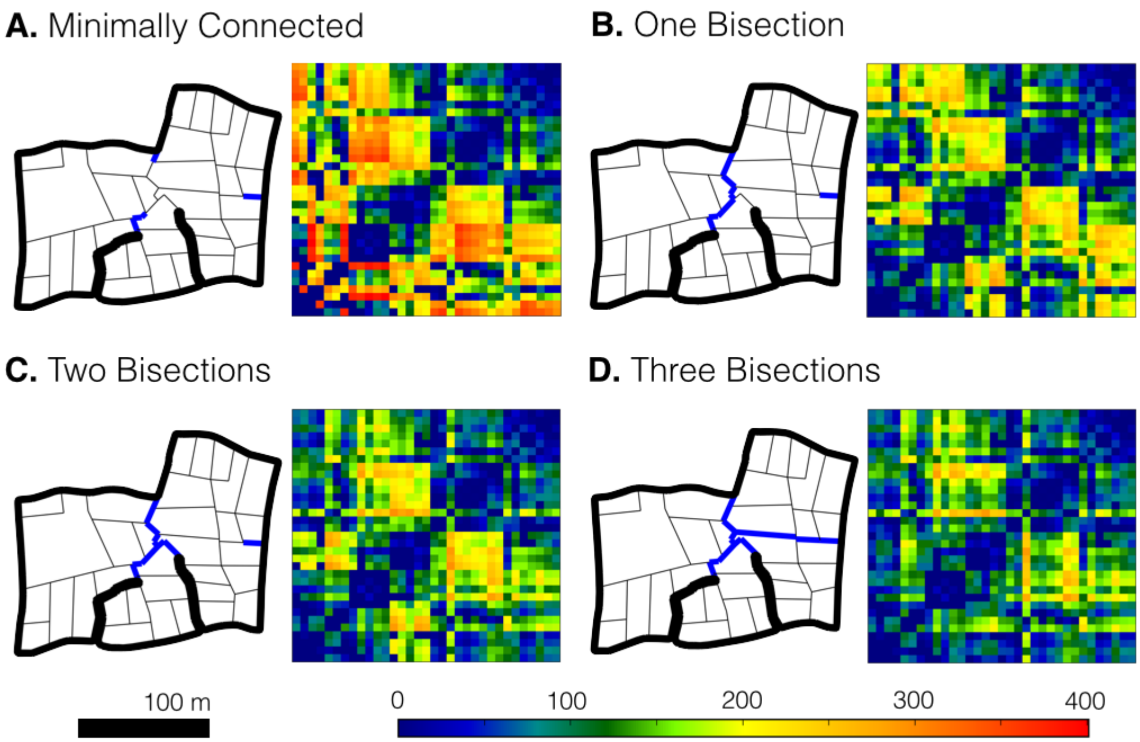


Figure S7: Epworth: Geometric Optimization and Travel Cost Matrices
A The minimal re-blocking strategy for the Epworth block of Figs. 1D, S4-5. The right panel matrices show travel distance (in meters) between any two parcels in the block after the introduction of the three cul-de-sacs (blue lines) that render the block universally accessible. Though all parcels are now accessible by roads, some remain distant to many others (red in the travel cost matrix). Such distances can be reduced through the introduction of more roads (see geometric optimization algorithm, Section C), leading to a non-minimal re-blocking strategy. **B** shows the effect of connecting two of the cul-de-sacs through block bisection. This leads to a substantial reduction of the average distance between parcels (35 m) and of the maximal distance between parcels, from 401 m to 205 m. Unlike the change in block topology that results from providing access to all parcels, which is discrete, the process of geometric optimization of travel distances is gradual. Thus, this second process of introducing more roads and reducing inter-parcel distances has no clear stopping point and could be continued until all boundaries become roads. **C** and **D** show how inter-parcel distances can be further reduced by additional road construction, especially due to extending cul-de-sacs to further bisect the resulting blocks.

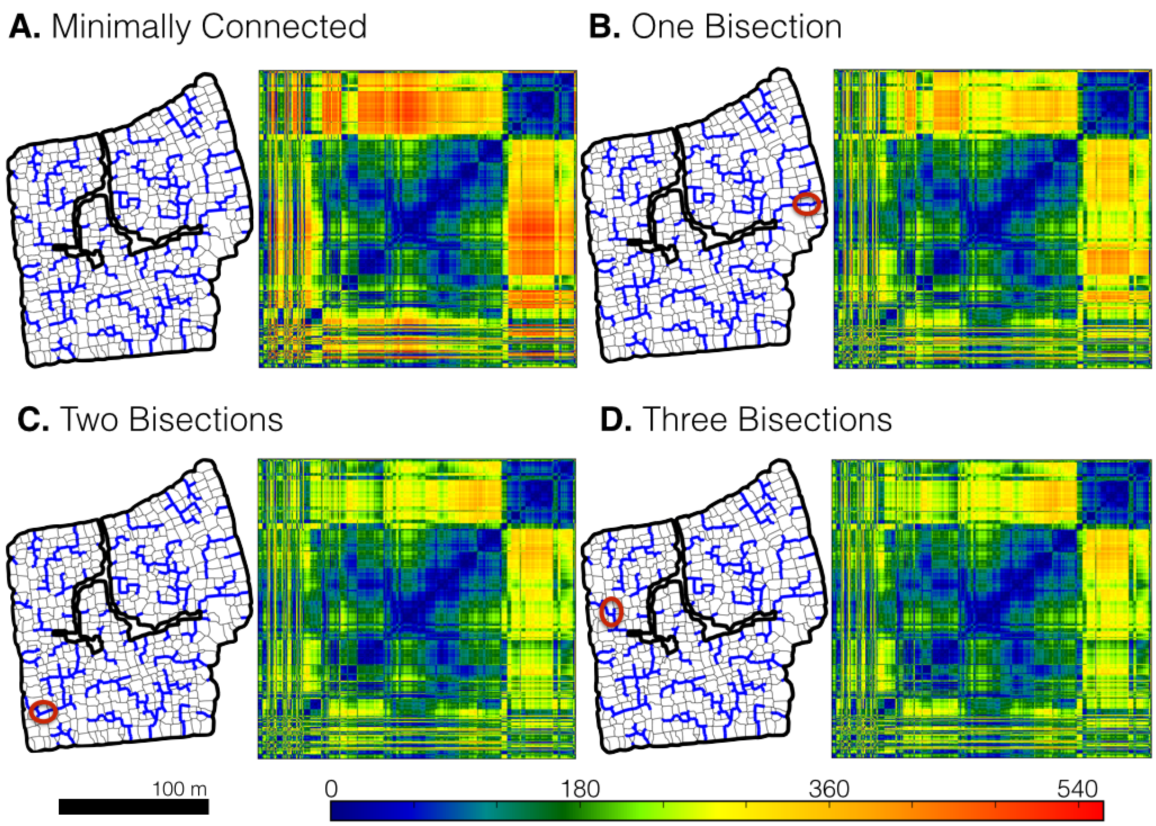


Figure S8. Khayelitsha: Re-blocking and Travel Cost Matrices
Mathematical methods become essential to upgrade large and dense slums, as in this example in Khayelitsha, a neighborhood of Cape Town, South Africa(*38*). There are 547 structures identified in this map, of which 396 do not have direct road access, for a population estimate of about 2,000 people. As in Fig. S7, **A** shows the road geometry and layout for the minimally re-blocked case and the associated parcel to parcel travel cost matrix. **B** The parcel-to-parcel travel cost matrix when a new road segment is added (circled in red). This leads to a significant reduction in average travel cost (27 m) in response to the addition of only a few meters of new road construction. C and D show how inter-parcel travel distances can be further reduced by further road construction.

Figure S9. Neighborhood Evolution. The historic evolution aof the dense old neighborhood (*115*) of San Antolin y San Marcos, Toledo, Spain. For a discussion in the broader context of Mediterranean urbanism, see Hakim (*50*). Similar spatial features are presented in many other studies e.g. in Fig. 33 in Hillier (*63*) and Fig. 1 in Buhl et. Al (*52*). See also Kostof and Tobias (*116*) for a general visual history of neighborhood spatial forms and urban transformations.

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