The Markov Process Behind Limited Information Trading

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I. Introduction:

Gode and Sunder (93) show that agents randomly bidding and selling in a double auction (each agent constrained not to lose money) will extract almost 100% of the surplus in the market. The trading process furthermore converges to the competitive equilibrium. Due to the limited optimization ability of their agents, the results of Gode and Sunder respond to the criticism that equilibrium theory assumes superhuman abilities of economic agents. The Gode-Sunder results were, however, for a partial equilibrium model. Hence, in evaluating of the importance of their results, it is necessary to ask whether the agents can be extended to a general equilibrium environment with the same results.

Putting randomly trading agents into an exchange economy should reveal whether the success of Gode and Sunder (93) was due in part to the simplicity of the task of the agents. In a small exchange economy, agents are required to trade more than one good simultaneously and operate as both buyer and seller. In Gode and Sunder (93) agents trade one good and work only one side of the market. I will show when randomly trading agents in a general equilibrium environment will or will not attain the competitive equilibrium.

Hurwicz, Radner and Reiter (75) have already done similar work with random agents. Hurwicz, Radner and Reiter (75) show that it is possible to trade to an efficient allocation in a multiple good economy with a fairly simple decentralized process. What is remarkable is that they prove agents, similar to those in Gode and Sunder(93), do not need complete information about other traders or the commodity set to reach an efficient allocation. Hurwicz, Radner and Reiter (75) do not however demonstrate that their process reaches the competitive equilibrium when it exists. The process presented in this paper is very close to the Hurwicz, Radner and Reiter process. My results show that the process put forward by Hurwicz, Radner and Reiter will not always reach the competitive equilibrium when it exists.

I will show that the competitive equilibrium is reachable by random trading only under very limited conditions. Trading behavior can be predicted using a Markov transition matrix because the matrix delineates a unique path of highest probability for the trading. If the competitive equilibrium is within a small neighborhood of this path the traders will randomly reach the competitive equilibrium.

To demonstrate the process, I have included the results from a computational version of the model as an example. The example illustrates the behavior of the traders. It concretely connects the trading outcome from several endowment points with the competitive equilibrium of the model, revealing when the competitive equilibrium is reached and when it is missed. The example has results from two sets of preference assumptions for the agents: Cobb Douglas and perfect complements.

II. The Model

A. Assumptions about the Agents and the Environment:

As noted in the introduction my trading process is similar to the process developed by Hurwicz, Radner and Reiter (75). To facilitate comparison of the models, I have adopted notation based on theirs.

The Agents:

Definition: The set of Agents I={1,2}, indexed by i.

Definition: The two types of agents {A,B}. There is one agent of each type.
Each type of agent has a fixed utility function. The utility functions U_A and U_B are monotonic concave and continuous.

The Environment:

Definition: The set of goods G={1,2}, indexed by l. The quantity of goods available is in continuous not discrete units.

Definition: The environment is a 10 by 10 Edgeworth Box; each good the market has a maximum of 10 units.

Definition: The set of feasible trades for each agent is X_i where
$X_i \equiv \{x_i : x_i = < x^1, x^2 > \& x_i^l \geq 0 \& x_i^l \leq 10, \forall l\}$. The set of possible trades for each agent must be within the dimensions of the Edgeworth Box. Agents cannot trade to a negative amount of a good or get more than 10 units of any one good.

Definition: The set of globally feasible trades X_G. First let $x \in X \equiv X_1 x X_2$. Let $X_G \equiv \{x: x_1 + x_2 =< 10,10 >\}$ such that global feasibility requires market clearing for all trades. The globally feasible set is by construction also individually feasible for both agents because each x is from the cross product of the feasible sets of each agent.

Definition: Each trading session from start to finish is called a market; within a market are periods denoted by t. The final period is denoted by τ so $t=0,\ldots,\tau$. Where τ is assumed to be finite because the agents are expected to reach an ε small distance from the contract curve. In a continuous space there is a zero probability that agents will reach any one allocation on the contract curve in a finite amount of periods.

B. The Trading Process:

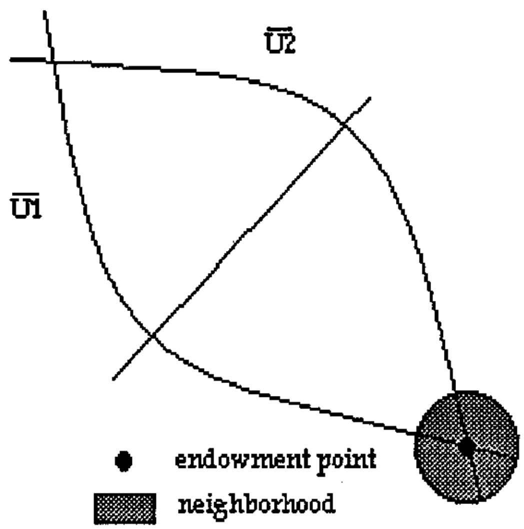
Let a market be a sequence of trades. At the beginning of each round an endowment x_i^o is given to each agent i. As defined above the trading takes place in periods. At each period t the agents also have an endowment point; let the endowment point be denoted by x^t_i. Each agent starts a period with an endowment and after trading ends each period with an allocation.

At each period t, the referee picks a point in the neighborhood of the endowment point and offers it to the agents as a possible allocation. The referee randomly pricks the offer using a uniform distribution over the neighborhood. Let the neighborhood at t be the set

$\psi(x^t) \equiv \{x: \|x_i^t - x_i\| \leq \delta, x_i \in X_i, \text{ for all i } \& \ x \in X_G\}$.

In the two agent case presented here, the neighborhood is a circle centered around the endowment point of the players. Distance as denoted by $\|Y\|$ refers to the points within the circle of radius δ from the endowment point. Offers made by the referee also must satisfy global feasibility conditions defined above. Figure 1 shows an example of a neighborhood.

Figure 1



Next, each agent must decide to accept or reject the offer made by the referee. Both traders must accept for a trade to be made. A trader accepts or rejects an offer based solely on whether the offer is in their contour set. The contour set for agent i is defined as the set

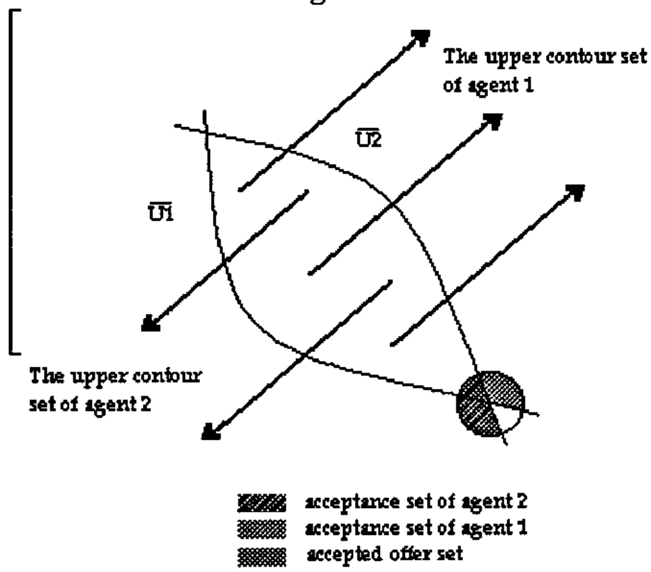
$\Omega(x_i^t) \equiv \{x_i: x_i \ \phi\!\approx x_i^t \ x_i \in X_i\}$.

The contour set is the set of possible trades that make the agent at least as well off as their allocation.
The contour set and the neighborhood then define the set of acceptable allocations for an agent. The acceptance set for an individual falls in the following intersection at t:

$A_i^t \equiv \psi(x^t) \cap \Omega_i(x_i^t)$, for i=1...n.

An accepted offer $a^t \in A_1^t \cap A_2^t$, comes from the intersection of the acceptance sets for both agents. Figure 2 demonstrates an acceptance set.

Figure 2



The accepted allocation will then be the endowment point of the next period such that $a^t = x^{t+}$. If the offer by the referee is rejected then the players keep their endowments and receive a new offer in the next period therefore $x^t = x^{t+1}$.

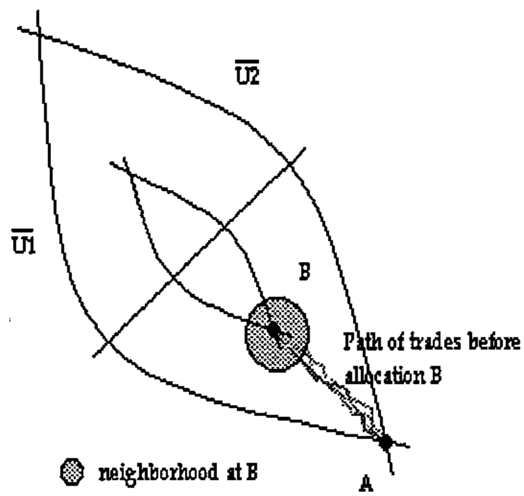
The trading is continued until the area $\Omega(x^t) \equiv X_f \cap \; X_{i=1}^{n} \Omega_i(x_i^t)$ collapses to an ε distance from the contract curve.

III. Results

The progression of trades across periods are generated by a continuous Markov process. It is continuous process because the state space of goods is continuous. The transition probability function for this process can be used to estimate a discrete matrix, Q, a graphical representation of the conditional probabilities of arriving at each point in the feasible set. The probability matrix Q reveals a highest probability trajectory over the space of trades. It is the highest probability trajectory that predicts if the traders will reach the Competitive Equilibrium.

The Markov process of this model is one with a continuous state space and with $\{x^t, 0 \le t < \infty\}$. The Markov transition function is $p(x|\theta) = \frac{1}{area(A^t)} I_{(A^t)}(x)$ where $A^t \equiv \mathop{X}_{i=1}^{n} A_i^t$, the Cartesian product of each individual's acceptance set at t. The vector of parameters θ includes the preferences, and endowment x^t. The function I(.) is an indicator function for points in the acceptance set. Given that the feasible set of trades within the boundaries of the indifference curves, $\Omega(x_i^0) \cap \Omega(x_2^0)$, is a Borel subset of the set of all trades within the Edgeworth Box, the probability measure $p(x|\theta)$ is a Markov transition function. To complete the definition of a Markov transition function, $p(x|\theta)$ is a Baire function for a fixed feasible set of trades and any time interval, s and t. The probability measure $p(x|\theta)$ also satisfies the Chapman-Kolmogorov equation (Doob, 53). Intuitively, the Markov property holds because given x^t for some t, transition probabilities are independent of past states. For example, given that the traders are at the endowment point x^t, the region $\Omega(x^t)$ will be the same no matter what path of allocations the agents traded in t=0,…,t-1. The example is demonstrated in Figure 3. The sequence of trades can take any path to reach point B and the neighborhood used by the referee to pick offers will remain the same.

Figure 3



To facilitate the description of the sequence of trades a simplification to a discrete approximation to the continuous Markov process would be instructive. For illustration purposes, let's consider a similar process that only has a discrete number of possible states. The state space is the discrete Cartesian product of the two agents' upper contour sets for the original endowment, x^o. Now let the matrix, Q contain the conditional probabilities of eventually passing through each point in the grid. Straightforward calculation will give each entry in Q. For a simple example take the discrete grid which is demonstrated in Table 1. The indifference curves that correspond to boundaries of this grid are from Cobb Douglas preferences.

Table 1

Start												
	1	0.333	0.111	0.037	0	0	0	0	0	0	0	0
	0.333	0.556	0.333	0.16	0.066	0.022	0.007	0.002	0	0	0	
	0.111	0.333	0.407	0.3	0.176	0.088	0.039	0.016	0.006	0.002	0	0
	0.037	0.16	0.3	0.336	0.271	0.178	0.102	0.052	0.025	0.011	0	0
	0	0.066	0.176	0.271	0.292	0.247	0.176	0.11	0.062	0.033	0.015	0
	0	0.022	0.088	0.178	0.247	0.262	0.228	0.171	0.114	0.07	0	0
	0	0.007	0.039	0.102	0.176	0.228	0.24	0.213	0.166	0	0	0
	0	0.002	0.016	0.052	0.11	0.171	0.213	0.222	0	0	0	0
	0	0	0.006	0.025	0.062	0.114	0.166	0	0	0	0	0
	0	0	0.002	0.011	0.033	0.07	0	0	0	0	0	0
	0	0	0	0.004	0.016	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
				Trajectory of Highest Probabilities								
				Absorption States								

The cells have the probability of getting to a trade given that the agents have started at the endowment in the upper left corner. The neighborhood around the endowment is fixed to be one unit. The one unit neighborhood means the traders start by moving to the right, downward, or diagonally down to the right. The probability associated with moving through the grid of possible actions sets up a probability of moving to any point which remains stationary throughout the process

There is a uniform probability that the agents will trade to any point in the neighborhood at a period t. The reasons that the agents move in this manor is that they always trade to a allocation at least as good as the one before it and the utility functions of the agents are all monotonic and therefore increasing in x. So there is a p=1/3 that any one point in picked at time t from the neighborhood of feasible allocations.

Also represented in the matrix in Table 1 is the absorption states of the contract curve. For simplicity the contract curve is represented by a line. In the continuous case, corresponding to the process of the model, the absorption line will not have the breaks in it that the discrete cases do. The points a small distance (ε) beyond the contract curve still will have a positive probability of being reached. If the traders cross the line then they will reverse the direction of their movement. Reversing direction is consistent with the rule that they cannot chose points that are not at least as good as the previous allocation. The discrete case shows the probability of getting to any point on the absorption line. In the infinite state case these probabilities are a continuous distribution over the contract curve.

The trajectory of highest probabilities will determine the mean of the distribution of over the contract curve. In Table 1 the trajectory can be traced from the starting endowment point through to the contract curve. The distance of the CE from the bisection point is a significant predictor of the success of the agents in getting to the maximum. If the competitive equilibrium is on or a small distance from the path of the highest probability the random traders will reach it. Only when this condition holds, will random agents in an Edgeworth Box optimize.

III. An Example:

Cobb Douglas Preferences:

The first set of results were for the following set of CES utility functions: $U_1=x_1^{\rho}+3y_1^{\rho}$ and $U_2=2x_2^{\rho}+y_2^{\rho}$, where $\rho = .000001$. The sets of endowment points were: {(2,2),(8,8)}, {(5,2),(5,8)}, {(8,2),(2,8)}, {(5,5),(5,5)}, {(8,5),(2,5)}, {(8,8),(2,2)}. This set was chosen to cover half of the area of the Edgeworth Box.

To get an idea of general trading behavior, it will be useful to look at the sequence of trades from start to finish. Figure 4 and 5 are paths for the first five completed markets for each set. Each graph places the trading in the initial set of indifference curves and shows the trades' relation to the contract curve.

Figure 4

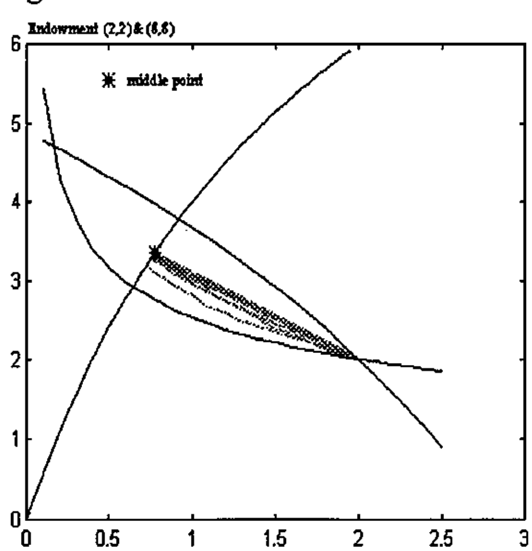
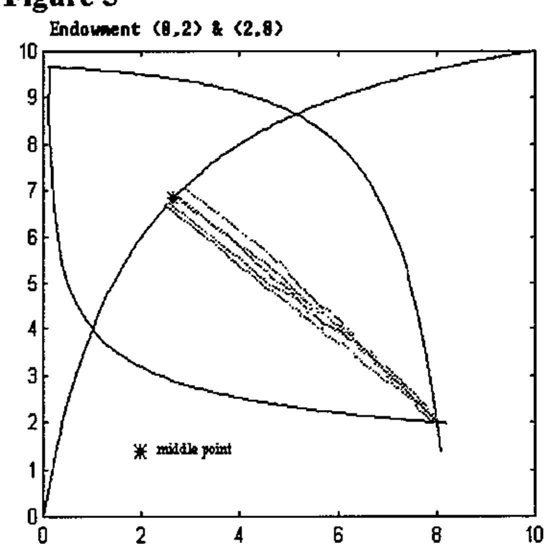


Figure 5



The lines that the sequences of trades make cluster together to produce a distribution of intersections with the contract curve. Also in the graphs is the middle of the contract curve between the two indifference curves. The mean sequence of trades for any given set of preferences and endowment points comes close to bisecting the contract curve within the indifference curves at the middle. The middle is therefore a rough estimate for where the final allocation of the process will be.

The results of the process will be close to the competitive equilibrium when the indifference curves are symmetric around the bisecting line of the ellipse. The mean sequence of trades intersect the contract curve near the middle and the middle overlaps with the competitive equilibrium in Cobb Douglas preferences when the curves are symmetric around the bisecting line of the ellipse.

The utility functions chosen create both symmetric and non-symmetric curves depending on the endowment point. Figures 6 shows a non-symmetric set of curves. Included are the competitive equilibrium and middle point. The scatter of points represent the final allocation for 30 runs of the trading. The points in Figure 6 are far from the competitive equilibrium. Contrast this to Figure 7 where the indifference curves are symmetric and the competitive equilibrium is near the middle of the contract curve. Here the scatter plot of final allocations are right over the competitive equilibrium.

Figure 6

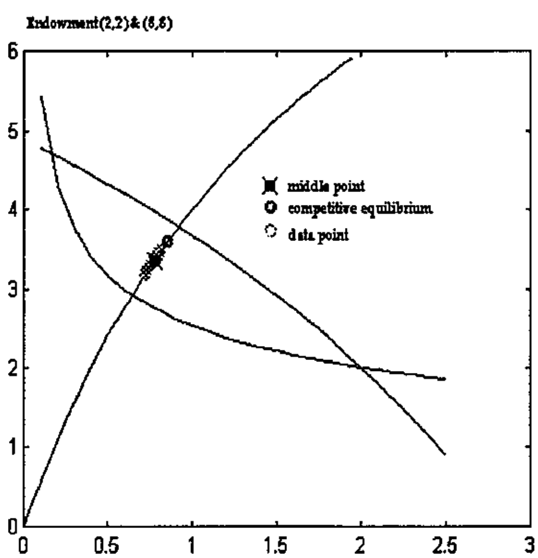
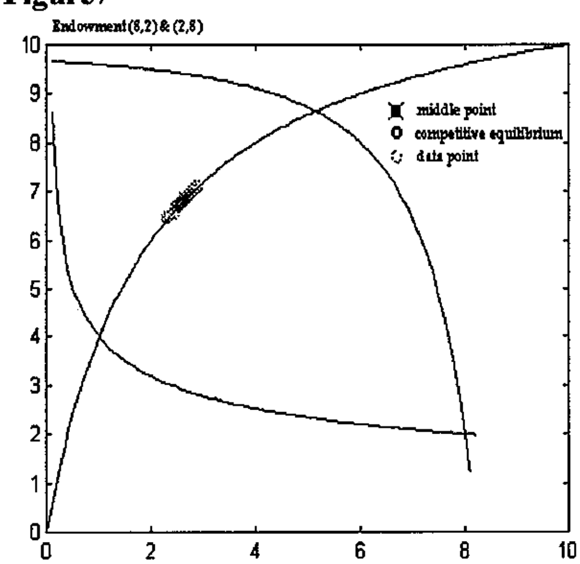


Figure7



T-tests were conducted on the differences between the competitive equilibrium price and the final prices and between the midpoint price and final prices for each set These t-statistics are presented in Table 2.

Table 2
t-statistics on Difference of Simulation Prices from:

	Middle Price	CE Price
Set 1	-3.591001	-23.58697
Set 2	-3.503675	-20.19512
Set 3	-1.411562	1.450362
Set 4	1.159837	-0.143283
Set 5	5.674977	19.93654
Set 6	5.452497	26.41997

Sets 3 and 4 both have symmetric indifference curves. The t-tests show their difference from the competitive price and the middle of the contract curve are not significantly different from zero. All the other sets have indifference curves that are not symmetric and they are not close to the competitive equilibrium.

These results show that market constraints are not enough to constrain trading in a two good economy. At each trade the indifference curves draw closer together but that does not translate into a constraint that eventually forces traders to get to the competitive equilibrium. It is interesting that random trading can get to the maximum at all but it is not the forces of the market that get the trades to the competitive equilibrium. Trading in the computational algorithm is a Markov process. It is the Markov transition matrix that constrains where the traders move. The transition matrix only relates to the market by the indifference curves that set its dimensions and the contract curve which creates a set of absorption states.

Leonteif Preferences:

The utility functions used were: $U_1=MIN(x_1,3y_1)$ and $U_2=MIN(2x_2,y_2)$. The sets of endowment points were different from those used for the Cobb Douglas preferences. With the Leonteif preferences it was more interesting to look at endowment points linearly related than to use points selected over the whole range of the Edgeworth Box. The endowment sets were: {(2,3),(8,7)}, {(4,3),(6,7)}, {(6.5,1.99),(3.5,8.01)}, {(7,1.99),(3,8.01)}, {(7.5,1.99),(2.5,8.01)}, {(8,1.99),(2,8.01)}, {(5,5),(5,5)}.

Figure 8 and 9 show that as the endowment point moves closer to the competitive equilibrium so do the final allocations.

Figure 8

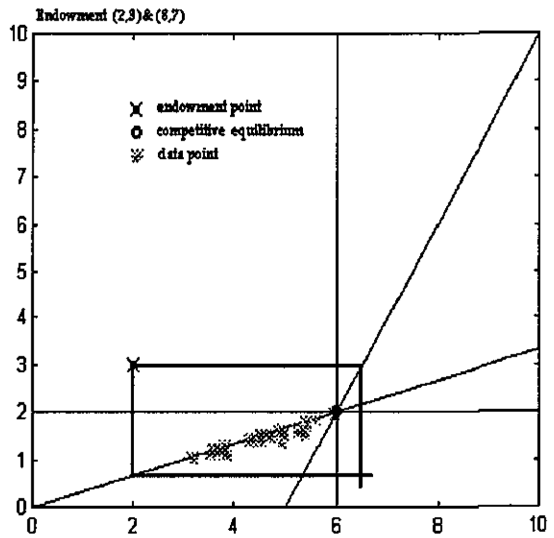
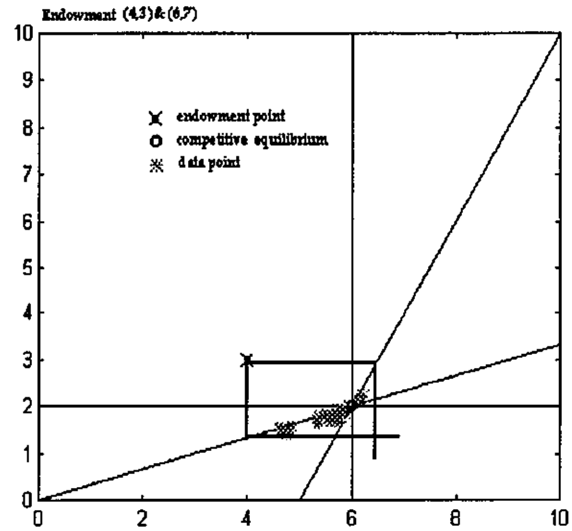


Figure 9



The distribution of data points becomes smaller as the area within the contract curve become smaller, but the data "cloud" will always lie at approximately the same angle relative to the endowment point. As in the case of Cobb-Douglas preferences, this angle will be dictated by the Markov Process guiding the trade movement. Yet when the competitive equilibrium is at a far corner of the space of feasible trades, as with both figures 8 and 9, it will be close to impossible to reach. The contraction of the indifference curves after every trade in the sequence will close the competitive equilibrium out of future trades. In Figures 8 and 9 the competitive equilibrium can only be reached with certainty when the player nearly indifferent between endowment and competitive equilibrium trades along their indifference curve.

When the competitive equilibrium is not in a corner of the feasible trade space, the angle of the endowment from the competitive equilibrium will dictate the success of the traders. This observation is directly related to the discussion of the mean sequence of trades in the Cobb Douglas section. If the competitive equilibrium is on the line of the mean sequence of trades for the Leonteif preferences the process will reach it.

IV. Conclusion:

The main conclusion to be made about the result is that the market constraints do not force the traders to the competitive equilibrium in an Edgeworth Box. It is not the competition between agents that guides the trades, but rather it is the random nature of the process. Traders wander down a random path that ends at the contract curve. Because the path is predictable the ability of the agents to reach a certain area on the contract curve is also predictable. The Gode and Sunder results can be explained similarly. In their case the market constraints were strict enough in relation to the random path of trading to force trading to the competitive equilibrium. It is as if the random traders are forced into a narrow canyon with the competitive equilibrium waiting by the only exit. So the random agents are consistently reaching the competitive equilibrium because it is inescapable. Without a narrow canyon the competitive equilibrium in a two good economy is more elusive.

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